Experiments with Linear and Semidefinite Relaxations for Solving the Minimum Graph Bisection Problem

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joint work with

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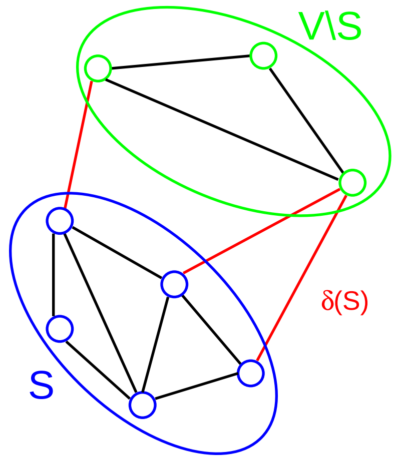
- Node Weighted Graph Bisection
- Knapsack Tree Inequalities
- Bisection Knapsack Walk Inequalities
- Connections to Odd Cycle Inequalities
- Strengthenings
- The Cluster Weight Polytope
- Some Numerical Results
- Conclusion and Problems

(supported by the DFG)

The Node Weighted Bisection Problem

- simple undirected Graph $G = (V, E)$, $V = \{1, \dots, n\}$, $E \subseteq \{ij : i, j \in V, i \neq j\}$ node weights $f_i \in \mathbb{N}_0$ for $i \in V$, capacity $F \in \mathbb{N}_0$,
- Find a bisection $(S, V \setminus S)$ with $f(S) := \sum_{i \in S} f_i \leq F$ and $f(V \setminus S) \leq F$ and $\delta(S)$ minimal (weights w_{ij})



$$P_B = \text{conv}\{\, y = \delta(S) : S \subseteq V,\ f(S) \leq F,\ f(V \setminus S) \leq F\} \qquad \subseteq P_{\text{CUT}}$$

IP-formulation: (suppose G contains a spanning star rooted at s)

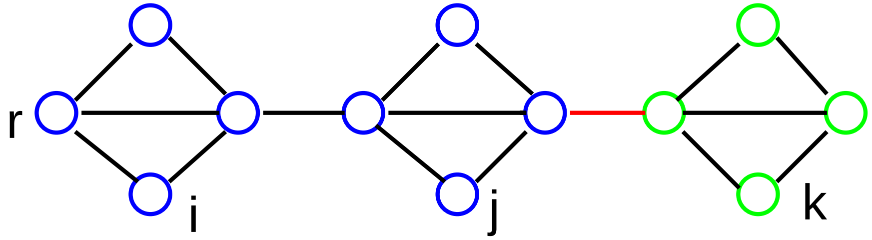
$$y_{ij} = \begin{cases} 1 & \text{if } ij \text{ is in the cut} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{lll} \text{min} & \sum_{ij} w_{ij} y_{ij} & \\ \text{s.t.} & f_s + \sum_{v \neq s} f_v (1 - y_{sv}) \leq F & i \in V \\ & \sum_{v \neq s} f_v y_{sv} \leq F & \\ & \sum_{ij \in C \setminus U} y_{ij} + \sum_{ij \in U} (1 - y_{ij}) \geq 1 & \text{cycle } C \subseteq E, \text{odd } U \subseteq C \\ & y \in \{0,1\}^E & \end{array}$$

Related Polyhedral Investigations/Surveys in the Literature

- Cut Polytope: [DezaLaurent97]
- Node Capacitated Graph Partitioning: [FMdSWW96]
- Equipartition: [ConfortiRaoSassano90], [deSouza93]
- Knapsack: [Weismantel97]

Bounding the size of the side belonging to some root $r \in V$



Given $y = \delta(S)$, does i belong to the same side as r?

Knapsack Tree Inequalities: [FMdSWW96]

Yes, if the shortest path P_{ri} from r to i consists of edges $y_e = 0$:

$$1 - \sum_{e \in P_{ri}} y_e \quad \text{positive} \quad \Rightarrow \quad i \text{ belongs to the same side as } r$$

$$\longrightarrow \qquad f_r + \sum_{i \in V_T \setminus \{r\}} f_i \left[1 - \sum_{e \in P_{ri}} y_e \right] \quad \leq \quad F$$

for some tree $T = (V_T, E_T)$ and its paths P_{ri} from r to i.

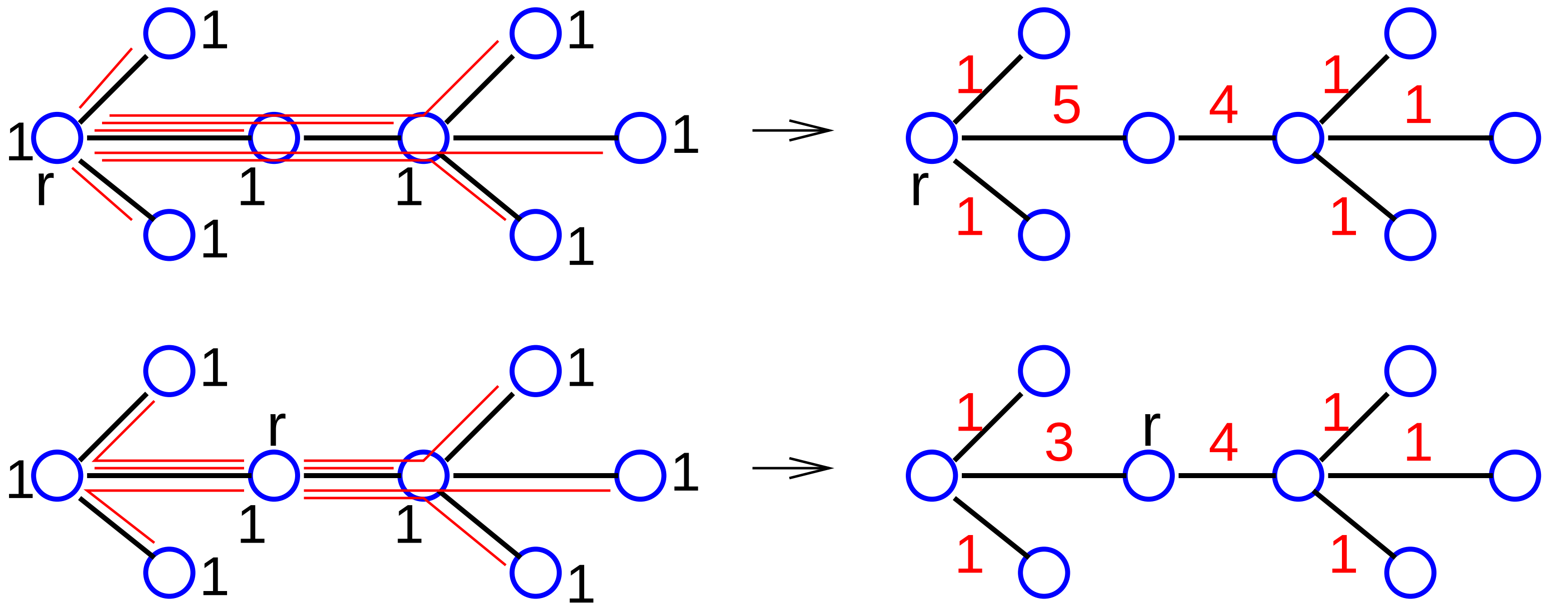
Collecting the weights of the edges along the paths from root r for each edge and applying "trivial" strengthening yields a *truncated knapsack tree ineq.*

If $\sum a_v z_v \leq a_0$ is valid for $P_K := \text{conv}\{z \in \{0,1\}^V : \sum_{v \in V} f_v z_v \leq F\}$ then

$$\sum_{e \in E_T} \alpha_e^r y_e \geq \alpha_0 \quad \text{with} \quad \alpha_0 := \sum_{v \in V_T} a_v - a_0 \quad \text{and} \quad \alpha_e^r := \min\{ \sum_{v : e \in P_{rv}} a_v, \alpha_0 \}$$

is valid for P_B. [FMdSWW96]

The Choice of the Root in Knapsack Tree Inequalities



Knapsack Tree Inequalities for a tree $T = (V_T, E_T)$

If $\sum a_v z_v \le a_0$ is valid for $P_K := \text{conv}\{z \in \{0,1\}^V : \sum_{v \in V} f_v z_v \le F\}$ then

$$\sum_{e \in E_T} \alpha_e^r y_e \ge \alpha_0 \quad \text{with} \quad \alpha_0 := \sum_{v \in V_T} a_v - a_0 \quad \text{and} \quad \alpha_e^r := \min\{\sum_{v : e \in P_{rv}} a_v, \alpha_0\}$$

is valid for P_B.

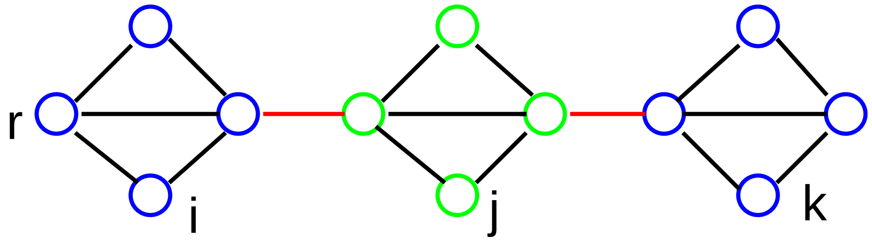
Best root in T is a *minimal root* $r \in \mathcal{R} = \underset{s \in V_T}{\text{Argmin}} \sum_{e \in E_T} \alpha_e^s$ (easy to find)

Theorem 1 *If $r \in \mathcal{R}$ then for all $s \in V_T$, $y \in P_B$* $\sum_{e \in E_T} \alpha_e^s y_e \ge \sum_{e \in E_T} \alpha_e^r y_e \ge \alpha_0$

For $f_v = 1$ $(v \in V)$ and root r each unreduced edge has weight $|V_e^r| = |\{v : e \in P_{rv}\}|$ (nodes "below" e) $\to$ minimal roots are "centered"

Theorem 2 *Assume $G = (V, E)$ is a tree, $f_v = 1$ $(v \in V)$, $\frac{|V|}{2} + 1 \le F \le |V|$, and for root r all edges of branchless paths have reduced knapsack weight. $\sum_{e \in E} \min\{|V_e^r|, |V| - F\} y_e \ge |V| - F$ is facet-defining iff r is a minimal root.*

Bounding the size of the side belonging to some root $r \in V$



Given $y = \delta(S)$, does i belong to the same side as r?

Knapsack Tree Inequalities:
Yes, if the shortest path P_{ri} from r to i consists of edges $y_e = 0$:

$$1 - \sum_{e \in P_{ri}} y_e \quad \text{positive} \quad \Rightarrow \quad i \text{ belongs to the same side as } r$$

$$\longrightarrow \qquad f_r + \sum_{i \in V_T \setminus \{r\}} f_i \left[1 - \sum_{e \in P_{ri}} y_e \right] \quad \leq \quad F$$

Bisection Knapsack Walk Inequalities: (exploit bipartition)
Yes, if there is a path P_{ri} with an <u>even</u> set $H_{ri} \subseteq P_{ri}$ of cut edges:

$$1 - \sum_{e \in P_{ri} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e) \quad \text{positive} \quad \Rightarrow \quad i \text{ belongs to the same side as } r$$

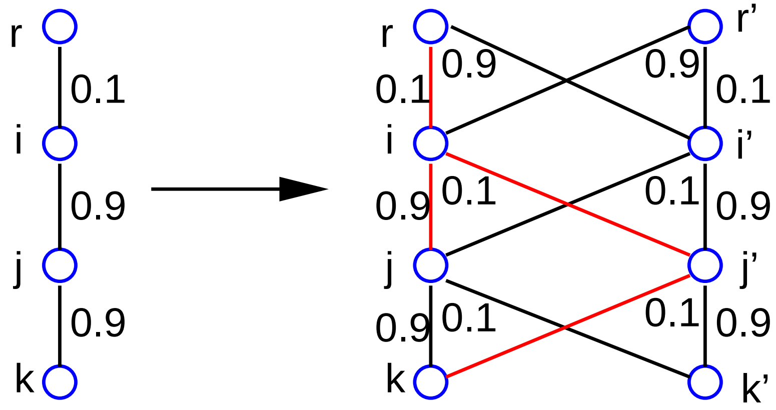
$$f_r + \sum_{i \in V \setminus \{r\}} f_i \left[1 - \sum_{e \in P_{rv} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e) \right] \quad \leq \quad F$$

Finding the best path P_{ri} with even set $H_{ri} \subseteq P_{ri}$

For $y \in [0,1]^E$, root r and node i the goal is to maximize

$$1 - \sum_{e \in P_{ri} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e)$$

simultaneously for all i by a shortest path tree in an auxiliary graph:



Note: • best *walk* P_{ri} and H_{ri} can be found in polynomial time
• they do not depend on the knapsack inequality $f(S) \leq F$
$\rightarrow$ find paths first, then use knapsack separator $\rightarrow a(S) \leq a_0$

Alg. almost identical to odd cycle separation for P_{CUT} [BM86]

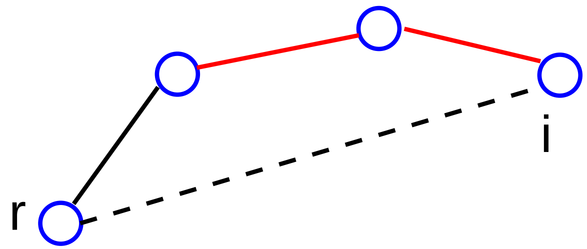
Connections of odd cycle to bisection knapsack walk inequalities

Odd cycle inequalities: Each cycle must be cut an even number of times
For C a cycle and odd $U \subseteq C$: $\sum_{e \in C \setminus U} y_e + \sum_{e \in U}(1 - y_e) \geq 1$

Suppose $y \in [0,1]^E$ and $ri \in E$, then i belongs to the same side as r if

$$1 - y_{ir} \qquad \text{is close to one.}$$

Let P_{ri} with even H_{ri} be the best bisection path from r to i with $ri \notin P_{ri}$,



Set $C = P_{ri} \cup \{ri\}$ and $U = H_{ri} \cup \{ri\}$ (odd) for the odd cycle inequality

$$1 - y_{ir} \;\; \geq \;\; 1 - \sum_{e \in P_{ri} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e)$$

- In the presence of odd cycle ineqs. a direct edge gives the best bound!
- Without a direct edge, one may derive the bound using odd cycle ineqs.

Theorem 3 *Let $G = K_n$ and y in the metric polytope, then the strongest bisection knapsack walk inequalities are stars.*

Strengthenings of Bisection Knapsack Walk Inequalities

- Trivial strengthening by rounding down coefficients that are too large

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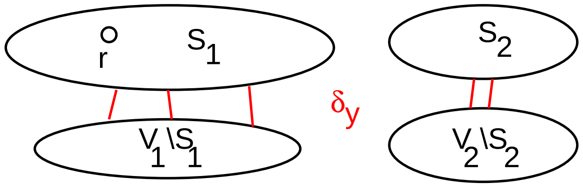
- If it is not possible/worthwhile to reach a part of G from root r:



Consider, e.g., a bisection cut y and connected components V_1 and V_2 with partitions $(S_1(y), V_1 \backslash S_1(y))$, $S_1(y) \subseteq V_1$ and $(S_2(y), V_2 \backslash S_2(y))$, $S_2(y) \subseteq V_2$, then

$$\sum_{i \in S_1(y)} f_i \leq F - \min\{\sum_{i \in S_2(y)} f_i, \sum_{i \in V_2 \backslash S_2(y)} f_i\}$$

$\rightarrow$ try to bound $\min\{\sum_{i \in S_2(y)} f_i, \sum_{i \in V_2 \backslash S_2(y)} f_i\}$ from below

$\rightarrow$ Study, for the knapsack inequality $a^T x \leq a_0$, $a \geq 0$, and subgraph $\bar{G}$ the polyhedron of the convex lower envelope of the function

$$\beta_{\bar{G}}(y) = \inf\{a(S), a(\bar{V} \setminus S) : S \subseteq \bar{V}, \max\{a(S), a(\bar{V} \setminus S)\} \leq a_0, y = \chi^{\delta_{\bar{G}}(S)}\}$$

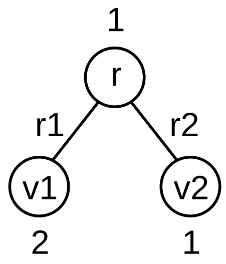
The Cluster Weight Polytope

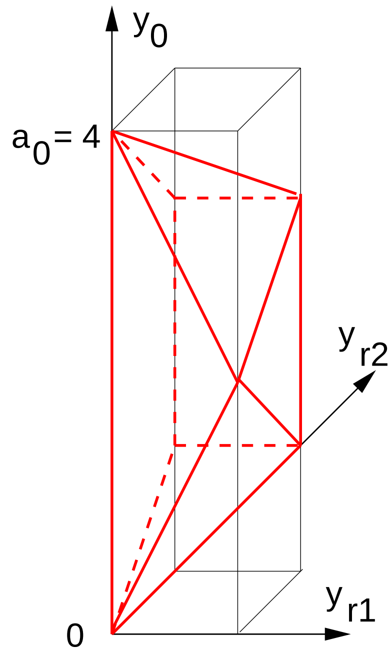
Definition:
For $G(V,E)$ and $a \in \mathbb{R}_+^V$, $a_0 \in \mathbb{R}_+$, set for $S \subseteq V$

$$h(S) = \begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix} \qquad =: \begin{pmatrix} y_0 \\ y \end{pmatrix}$$

then the cluster weight polytope is the set

$$P_{CW} = \text{conv}\,\{h(S) :\ S \subseteq V,\ a(S) \le a_0,\ a(V \backslash S) \le a_0\}$$





Observe that for feasible S,

$$\frac{1}{2}h(S) + \frac{1}{2}h(V \setminus S) = \begin{pmatrix} \frac{1}{2}a(V) \\ \chi^{\delta(S)} \end{pmatrix}$$

Observation 4 *P_{CW} is symmetric with respect to the hyperplane $y_0 = \frac{1}{2}a(V)$*

We will give a complete description of P_{CW} for stars with $a_0 \ge a(V)$.
(Stars can be separated reasonably well within the bisection setting)

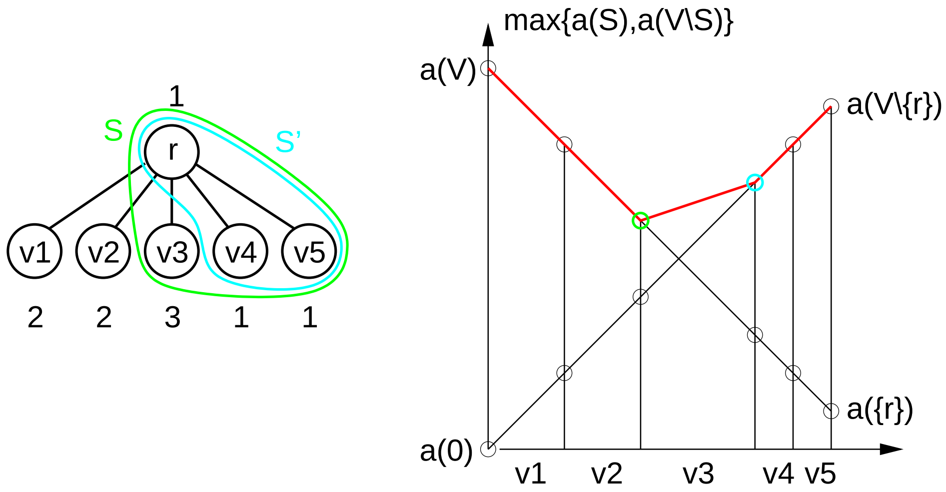
The nontrivial facets of P_{CW} for stars with $a_0 \geq a(V)$

Call a triple $(V_p, \bar{v}, V_n)$ *feasible* if it is a partition of $V \setminus \{r\}$ with

$$a(V_p) \leq \frac{1}{2}a(V) \qquad \text{and (if } \bar{v} \text{ exists)} \qquad a(V_p \cup \{\bar{v}\}) > \frac{1}{2}a(V).$$

For each feasible triple the following inequality is facet defining:

$$y_0 + \sum_{i \in V_p} a_i y_{ri} + [a(V) - 2a(V_p) - a_{\bar{v}}]\, y_{r\bar{v}} - \sum_{i \in V_n} a_i y_{ri} \leq a(V)$$



Its symmetric version is

$$y_0 - \sum_{i \in V_p} a_i y_{ri} - [a(V) - 2a(V_p) - a_{\bar{v}}]\, y_{r\bar{v}} + \sum_{i \in V_n} a_i y_{ri} \geq 0$$

The complete description of P_{CW} for stars with $a_0 \geq a(V)$

Theorem 5 *$G = (V, E)$ a star with root $r \in V$, $a \in \mathbb{R}_+^E \setminus \{0\}$, $a_0 \geq a(V)$.*

- *If $a_r < \frac{1}{2}a(V)$ then P_{CW} is the set of points satisfying*

$$
\begin{array}{ll}
0 \leq y_{ri} \leq 1 & \forall ri \in E \\
y_0 + \sum_{i \in V_p} a_i y_{ri} + [a(V) - 2a(V_p) - a_{\bar{v}}]\, y_{r\bar{v}} - \sum_{i \in V_n} a_i y_{ri} \leq a(V) & \forall \text{feasible } (V_p, \bar{v}, V_n) \\
y_0 - \sum_{i \in V_p} a_i y_{ri} - [a(V) - 2a(V_p) - a_{\bar{v}}]\, y_{r\bar{v}} + \sum_{i \in V_n} a_i y_{ri} \geq 0 & \forall \text{feasible } (V_p, \bar{v}, V_n)
\end{array}
$$

- *If $a_r \geq \frac{1}{2}a(V)$ then P_{CW} is the set of points satisfying*

$$
\begin{array}{ll}
0 \leq y_{ri} \leq 1 & \forall ri \in E \\
y_0 + \sum_{i \in V \setminus \{r\}} a_i y_{ri} \leq a(V) & \\
y_0 - \sum_{i \in V \setminus \{r\}} a_i y_{ri} \geq 0 &
\end{array}
$$

The decisive step in the proof:

All inequalities bounding y_0 from above are of this form.

Sketch of proof: We use

Lemma 6 *Suppose* $y_0 + \sum_{i \in V \setminus \{r\}} \gamma_i y_{ri} \leq \gamma_0$ *is a facet of* P_{CW}. *Then*

$$-a_i \leq \gamma_i \leq a_i \qquad i \in V \setminus \{r\}$$

$$\gamma_0 = a(V)$$

$$\sum_{i \in V \setminus \{r\}} \gamma_i \leq a_r$$

For a given $\bar{y} \in [0,1]^E$ find the best inequality bounding y_0 by solving

$$\begin{array}{ll} \min & a(V) - \sum_{i \in V \setminus \{r\}} \bar{y}_{ri} \gamma_i \\ \text{s.t.} & \sum_{i \in V \setminus \{r\}} \gamma_i \leq a_r \\ & -a_i \leq \gamma_i \leq a_i \quad i \in V \setminus \{r\} \end{array} \quad \begin{array}{c} \xi_i := \gamma_i + a_i \\ \Longleftrightarrow \end{array} \quad \begin{array}{ll} \max & \sum_{i \in V \setminus \{r\}} \bar{y}_{ri} \xi_i \\ \text{s.t.} & \sum_{i \in V \setminus \{r\}} \xi_i \leq a(V) \\ & 0 \leq \xi_i \leq 2a_i \quad i \in V \setminus \{r\} \end{array}$$

Continuous knapsack with greedy solution: Set ξ_i to max sorted by $\bar{y}_{ri}$:

$\rightarrow \xi_i = 2a_i$ for $i \in V_p \subseteq V$ with $\min_{i \in V_p} \bar{y}_{ri} \geq \max_{j \in V \setminus (V_p \cup \{r\})} \bar{y}_{rj}$ and $\sum_{i \in V_p} 2a_i \leq a(V)$

and the fractional $\xi_{\bar{v}} = a(V) - 2a(V_p)$

$$\Rightarrow \qquad y_0 + \sum_{i \in V_p} a_i y_{ri} + [a(V) - 2a(V_p) - a_{\bar{v}}]\, y_{r\bar{v}} - \sum_{i \in V_n} a_i y_{ri} \leq a(V)$$

Finding Stars for Knapsack Walk Inequalities when $f_i = 1 \; \forall i$

Given $$\sum_{i \in V \setminus \{r\}} \pi_i := \sum_{i \in V \setminus \{r\}} \left[1 - \sum_{e \in P_{rv} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e) \right] \leq F - 1,$$

find the best star rooted at some $s \in V \setminus \{r\}$ to strengthen this inequality.

V_s ... candidate nodes for the star rooted at s (r excluded)
decide on $\bar{v}$, $|V_p| = |V_n|$ or do not include

Build the star by adding pairs of nodes, one to V_p and one to V_n with gain

$$|y_{si} - y_{sj}| - \pi_i - \pi_j$$

(and a pair consisting of the root s and a node j with gain $y_{sj} - \pi_j$)

$\rightarrow$ construct an auxiliary graph and find a maximum weight matching

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$\Rightarrow$ if $f_i = 1 \; \forall i$ the most violated star strengthened knapsack walk inequalities can be found in polynomial time.

The Setting of the Numerical Experiments

Used LP and SDP-relaxation in the same Branch&Cut-framework SCIP
[thanks to Tobias Achterberg from ZIB Berlin]

LP-relaxation

- basic relaxation: add a star into G if necessary and separate odd cycles
- solve LPs using CPLEX

SDP-relaxation

- use same graph as LP
- canonical max-cut relaxation in $\{-1, 1\}$-variables ($\mathrm{diag}(X) = e, X \succeq 0$)
- capacity constraint by $\left\langle ff^T, X \right\rangle \leq (2F - f(V))^2$,
- solve dual by Spectral Bundle Method with primal aggregation
- separate on primal aggregate w.r.t. the support, possibly enlarge the support

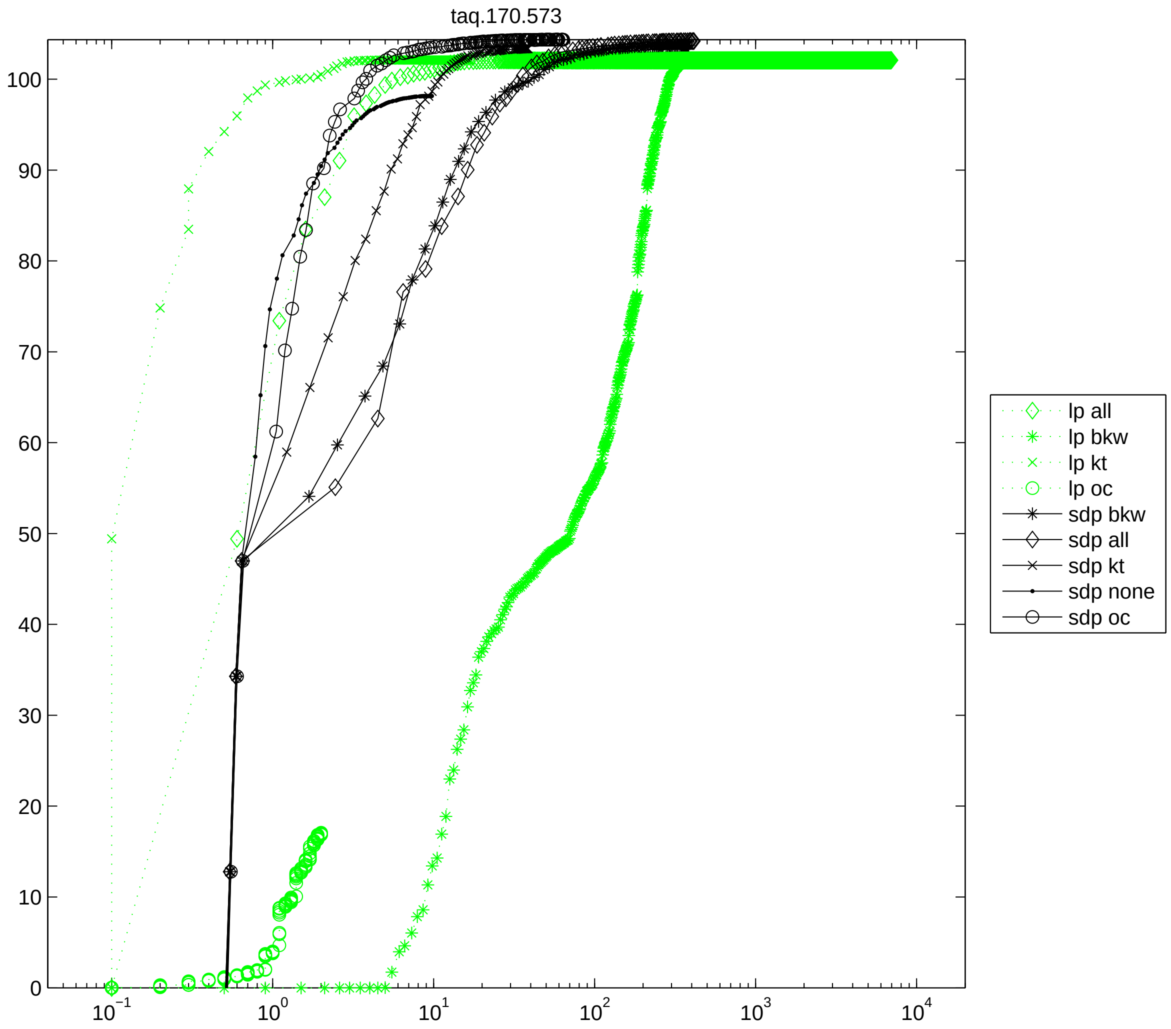
Separation routines

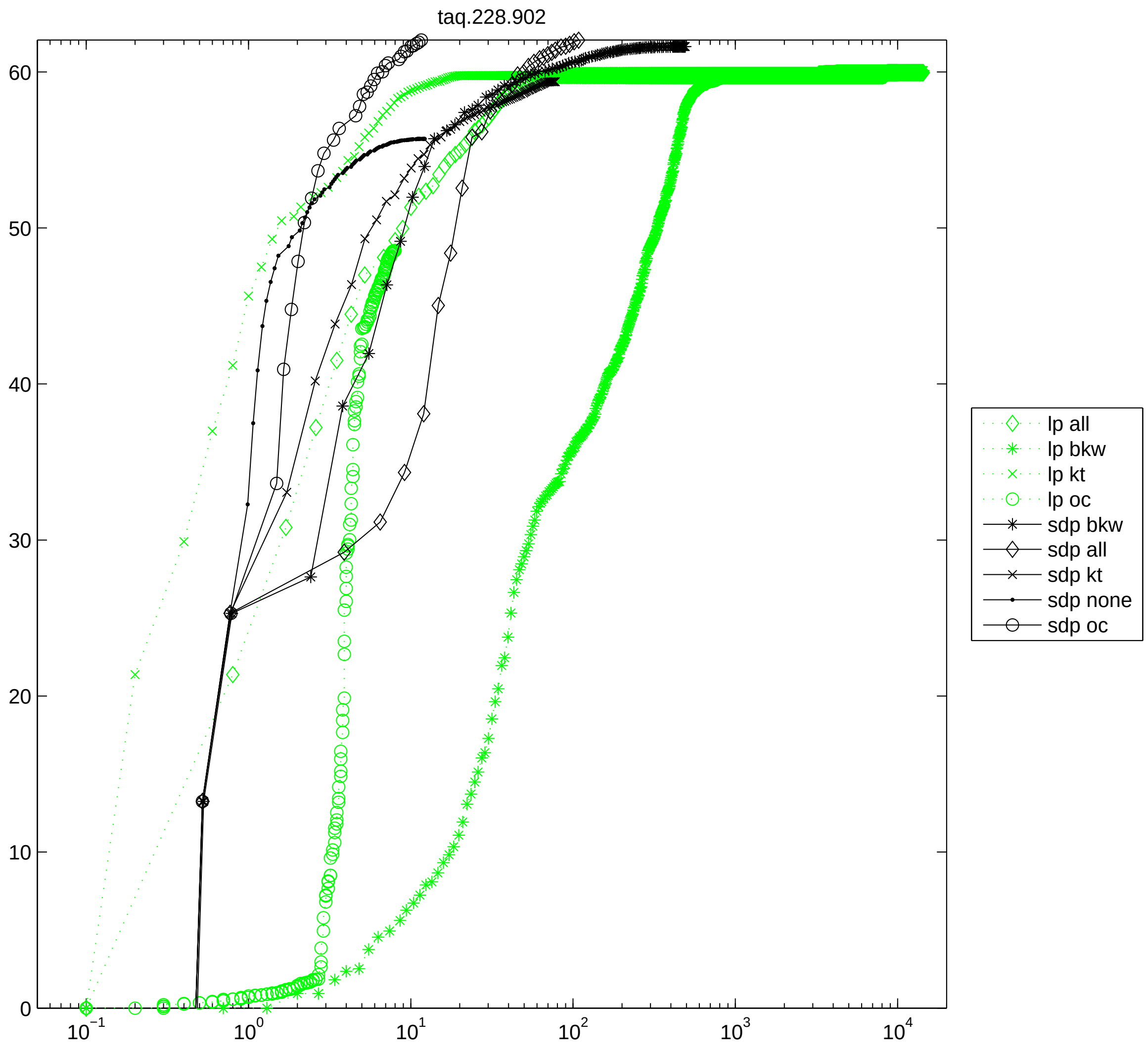
except for odd cycles, both use the same separation routines for
knapsack star and bisection knapsack walk inequalities

Root Node Value and Computation Time ($\leq 4h$)

graph.n.m	linear relaxation				semidefinite relaxation					pb
	oc	*oc+bkw*	*oc+kt*	*all*	*none*	*bkw*	*kt*	*oc*	*all*	
taq.170.573	17.1*	102.1	**102.1**	102.1	98.4	103.7	103.4	**104.4**	104.2	110
	$2s$	$567s$	$16s$	$6975s$	$14s$	$384s$	$39s$	$64s$	$413s$	
taq.228.902	48.5	**59.9**	59.7	59.7	55.7	61.6	59.4	**63.0**	63.0	63
	$8s$	$1461s$	$503s$	$4h$	$36s$	$493s$	$78s$	$12s$	$108s$	
diw.681.2152	18.4	53.4	**136.3**	135.5	79.7	135.5	135.8	**140.5**	138.1	142
	$197s$	$4h$	$4h$	$4h$	$4h$	$4h$	$4h$	$4h$	$4h$	
taq.1021.3259	23.1	49.5	**113.1**	102.8	60.2	112.9	113.9	**118.0**	116.3	118
	$667s$	$4h$	$4h$	$4h$	$78s$	$4h$	$4h$	$2\,147s$	$4h$	
taq.334.3952	75.5	287.1	**326.0**	326.4	235.9	325.4	322.1	**328.6**	324 .8	342
	$67s$	$4h$	$4h$	$4h$	$123s$	$1778s$	$246s$	$884s$	$6256s$	
dmxa.1755.5420	12.4	31.0	**83.2**	80.1	37.7	88.9	90.0	**94.0**	91.0	94
	$3159s$	$4h$	$4h$	$4h$	$178s$	$4h$	$4h$	$9016s$	$4h$	
taq.1021.6356	74.1	260.4	**703.9**	618.8	1125.4	1481.0	1470.1	**1578.7**	1504.2	1692
	$1036s$	$4h$	$4h$	$4h$	$125s$	$4h$	$4h$	$4h$	$4h$	
gap.2669.8841	8.6	33.2	**73.5**	72.5	35.9	71.9	73.3	**73.5**	72.5	74
	$4h$	$4h$	$4h$	$4h$	$585s$	$4h$	$4h$	$4h$	$4h$	

* lower bound, n number of nodes, m number of edges


taq.170.573
lp all
lp bkw
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc
0
10
20
30
40
50
60
70
80
90
100
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}


taq.228.902
0
10
20
30
40
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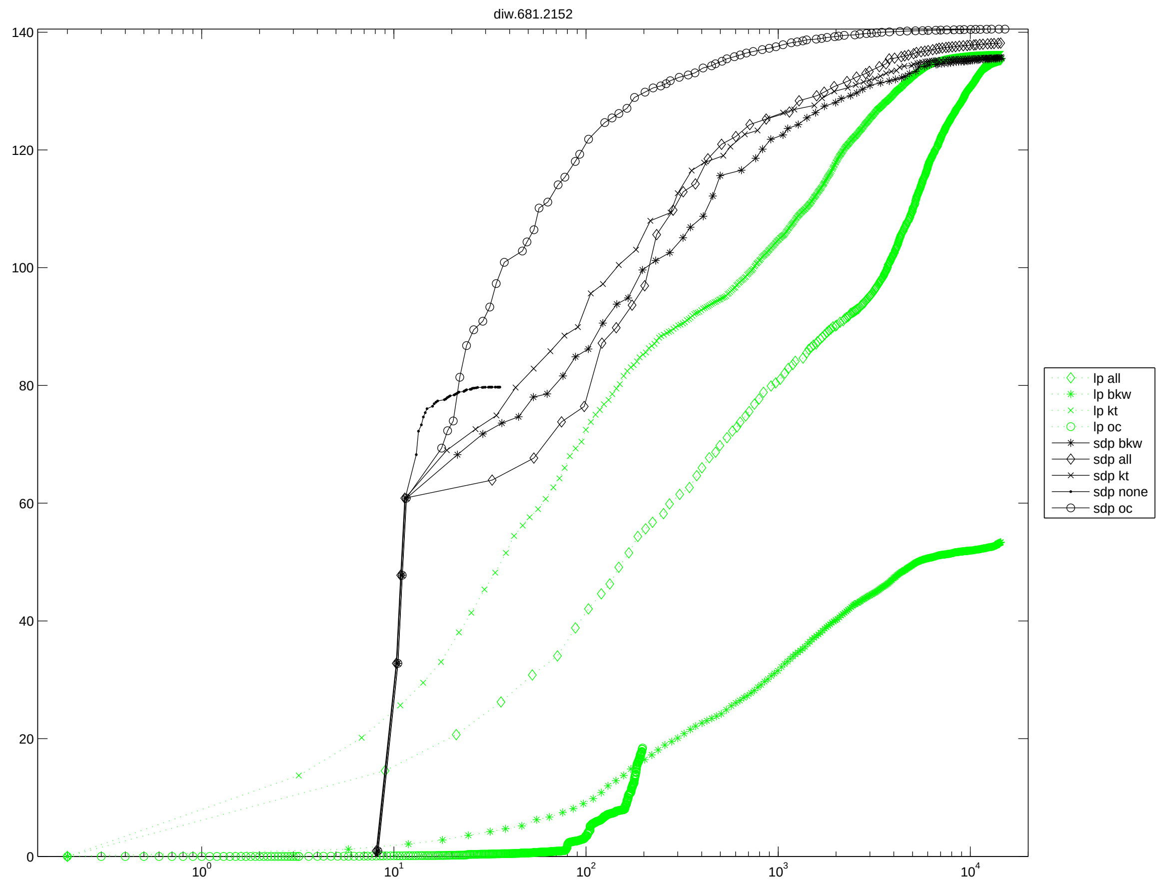
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
lp all
lp bkw
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc


diw.681.2152
140
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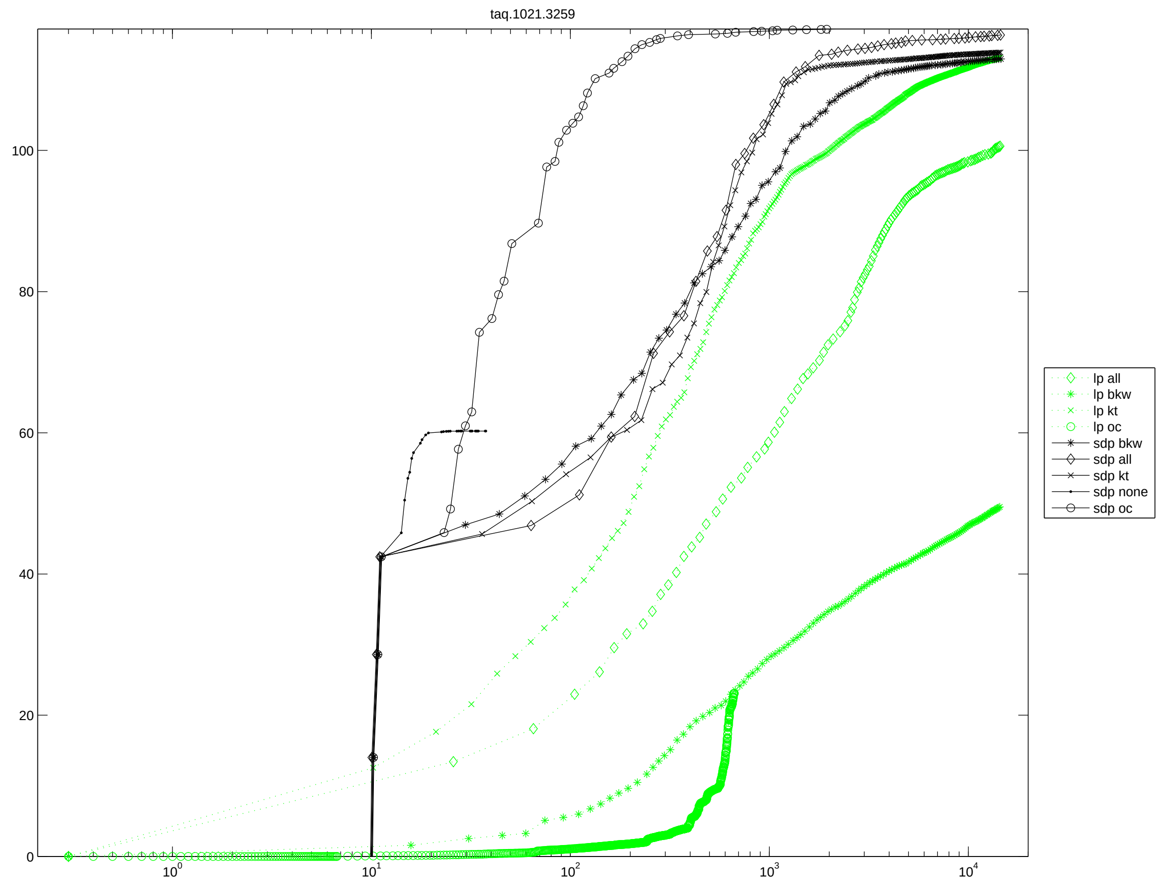
10^4
lp all
lp bkw
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc


taq.1021.3259
100
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lp all
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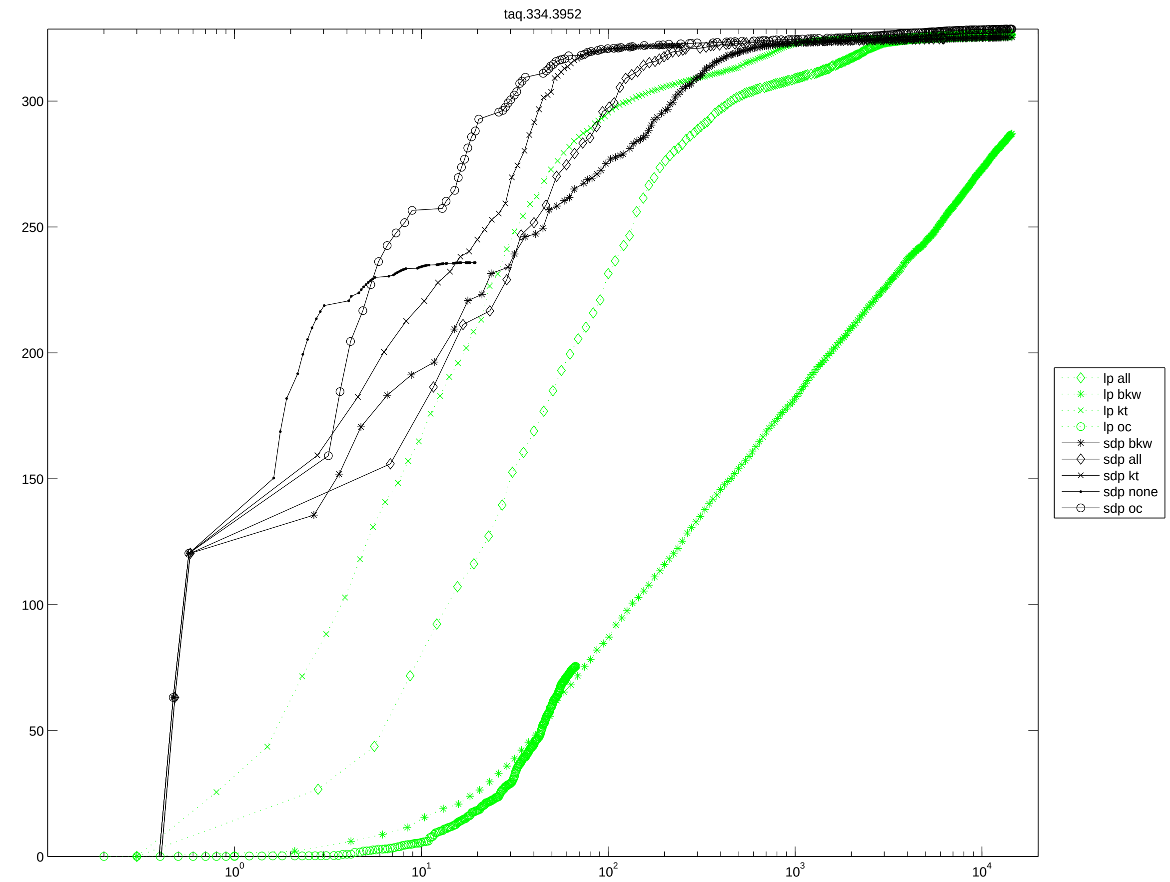
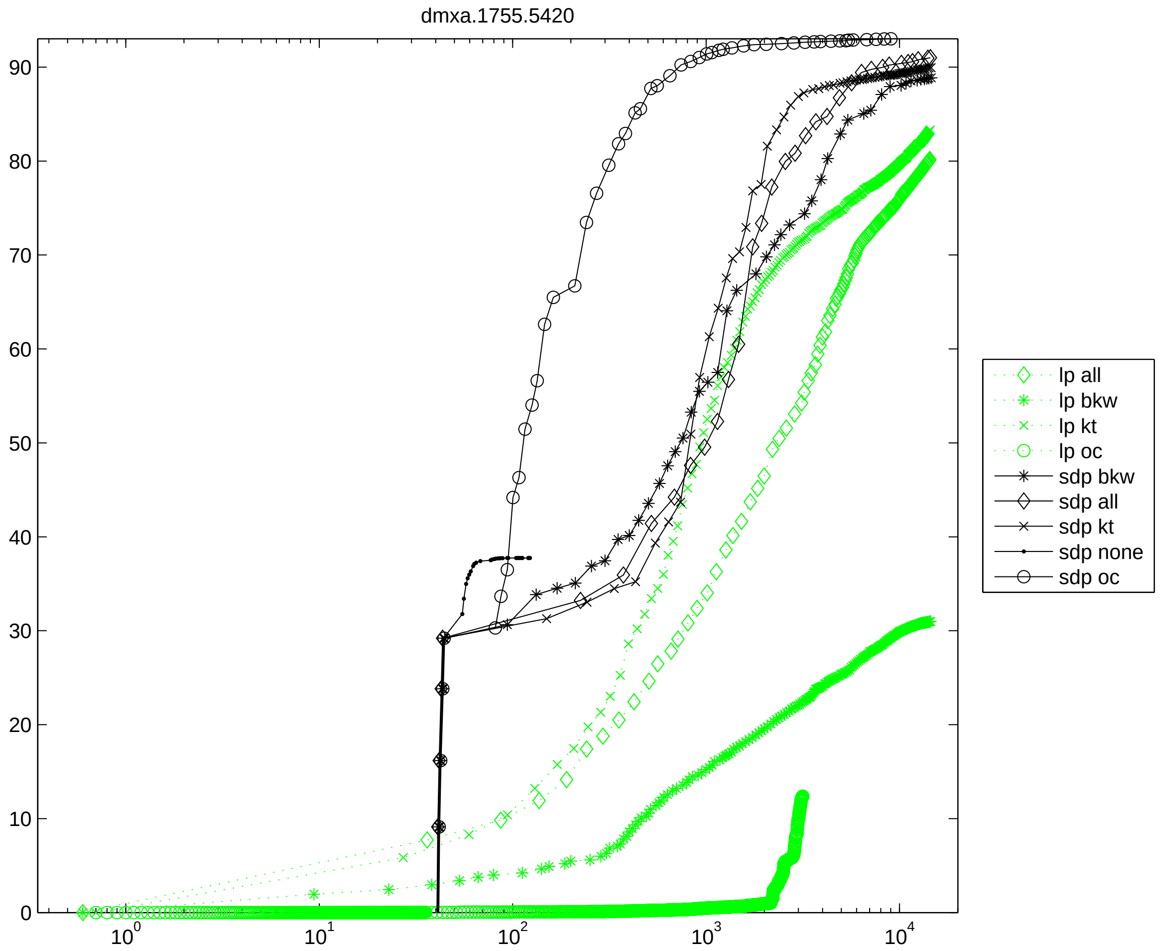
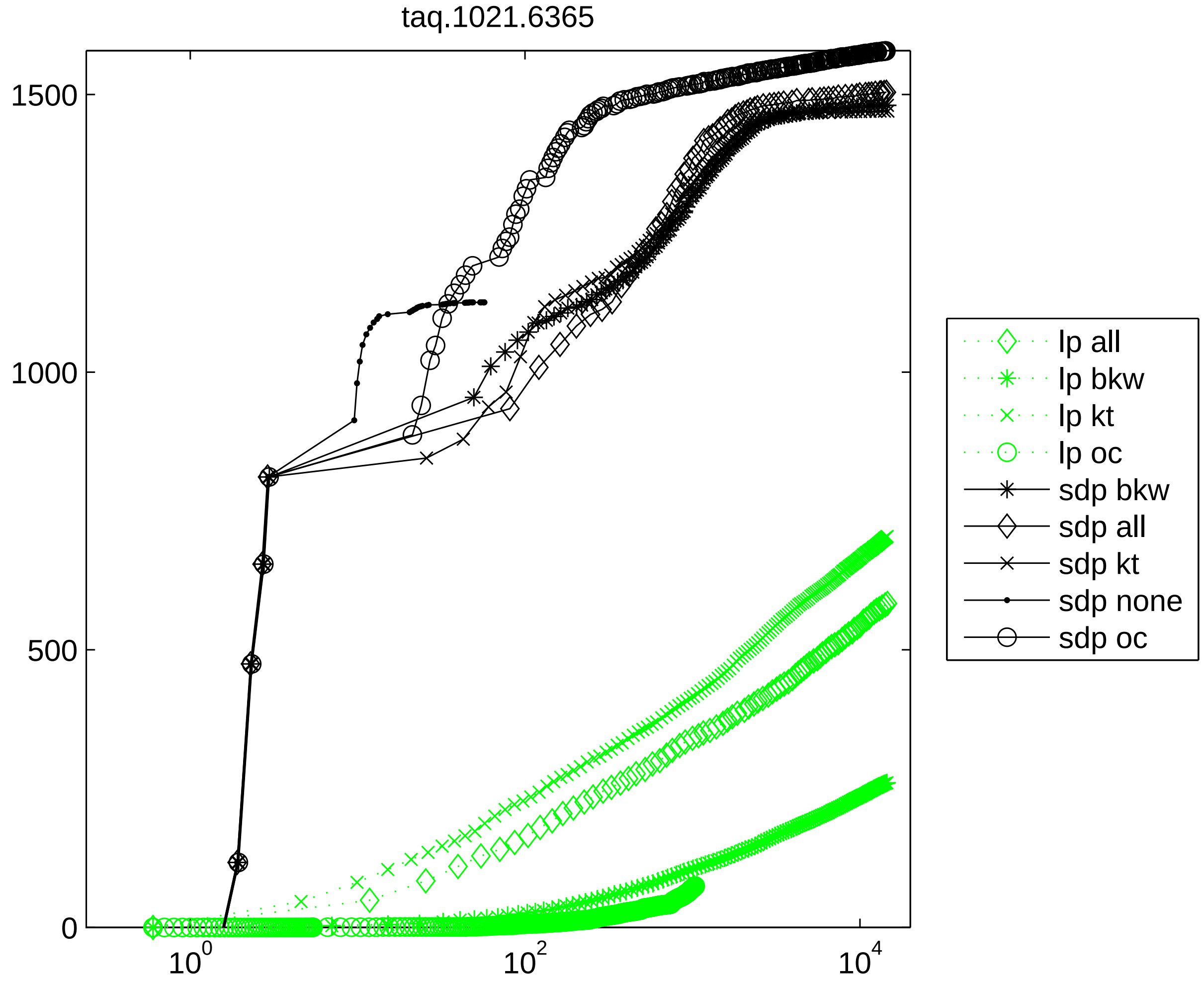
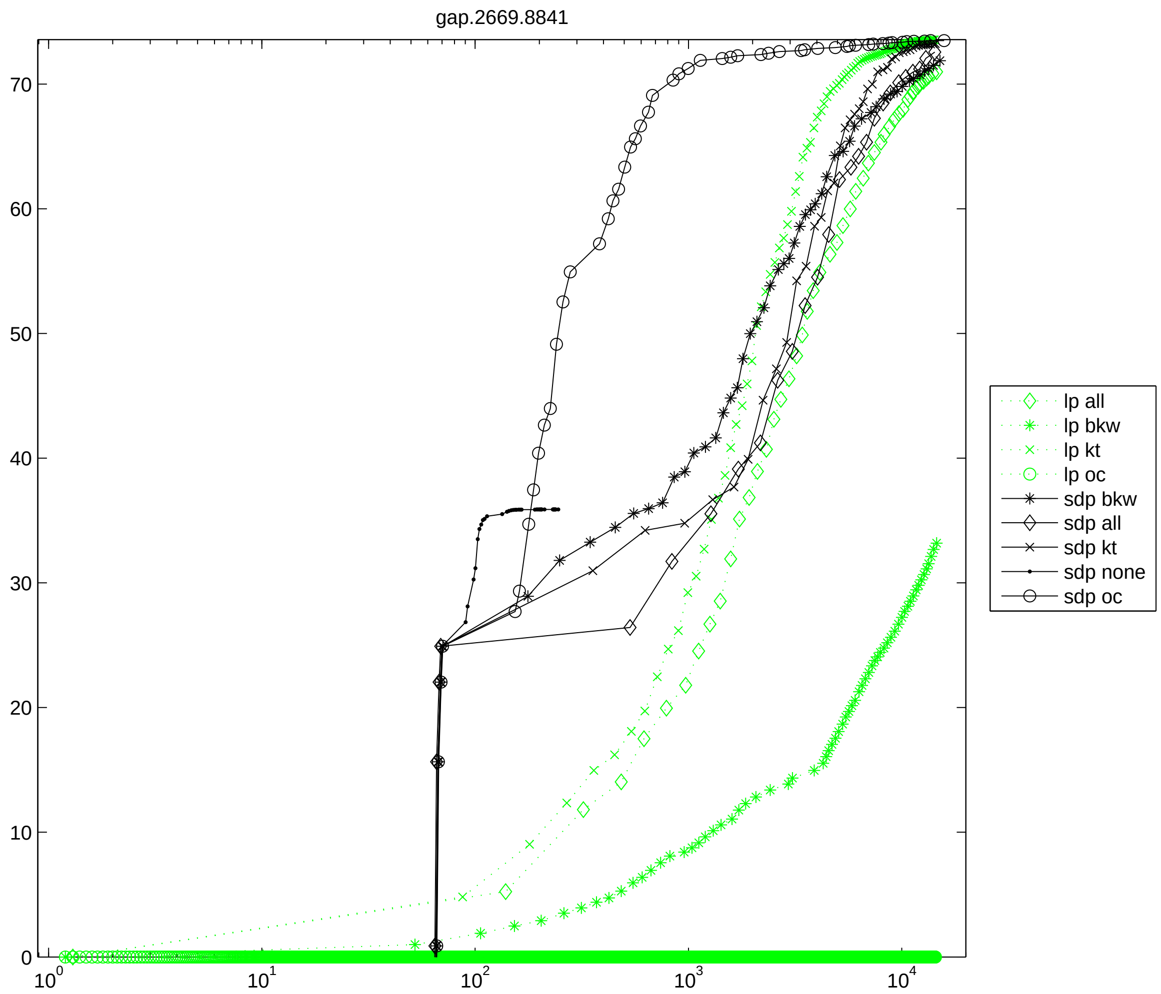
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc



taq.334.3952
300
250
200
150
100
50
0
10^0
10^1
10^2
10^3
10^4
lp all
lp bkw
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc


dmxa.1755.5420
0
10
20
30
40
50
60
70
80
90
10^0
10^1
10^2
10^3
10^4
lp all
lp bkw
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc


taq.1021.6365
1500
1000
500
0
10^0
10^2
10^4
lp all
lp bkw
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc


gap.2669.8841
0
10
20
30
40
50
60
70
10^0
10^1
10^2
10^3
10^4
lp all
lp bkw
lp kt
lp oc
sdp bkw
sdp all
sdp kt
sdp none
sdp oc

Linear vs. Semidefinite Relaxation - Branch & Cut

graph	#sec.	# bc nodes	primal bound	dual bound	gap (%)
linear relaxation					
taq.170.573	125	935	110	110.0	0
taq.228.902	70	13	63	63.0	0
diw.681.2152	14407	791	482	98.7	388
taq.1021.3259	14407	100	444	100.6	341
taq.334.3952	14403	885	342	331.5	3
dx.1755.5420	14404	14	1088	67.7	1506
taq.1021.6356	14401	14	4125	631.2	553
gap.2669.8841	14407	47	1810	63.1	2767
semidefinite relaxation					
taq.170.573	2670	7	110	110.0	0
taq.228.902	120	1	63	63.0	0
diw.681.2152	14596	5	142	137.1	3
taq.1021.3259	14767	4	118	115.3	2
taq.334.3952	14407	10	342	330.9	3
dx.1755.5420	15838	3	94	86.4	8
taq.1021.6356	14841	1	1787	1504.2	18
gap.2669.8841	14493	2	78	71.2	9

Summary

- In Knapsack Tree Ineqs. the root should be chosen carefully
- Knapsack Walk Ineqs. are a specialization of Knapsack Trees to Bisection, they are closely related to Odd Cycle Inqualities, the best paths from the root to each node can be found in polynomial time
- We introduced the Cluster Weight Polytope and gave its complete description for stars without capacity limit.
- For $f_v = 1$, Star Strengthened Bisection Knapsack Walk Inequalities can be separated in pol. time
- For general bisection problems, current SDP-relaxation approaches are competitive if not superior to current LP-techniques in practice

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Problems

- Complete description of the Cluster Weight Pol. for trees and $a_0 \geq a(V)$? (would allow to span entire components; no hope for $a_0 < a(V)$)
- Strengthen Knapsack Walk Inequalities by stars for general weights