Hyperbolic set covering problems with competing ground-set elements

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Outline

- Problems definition
- The motivating application: Wireless Local Area Network design
- Hyperbolic integer programming formulation
- Complexity and Approximability results
- Linearizations and Lagrangean Relaxation
- Ongoing work and concluding remarks

Set Covering notation

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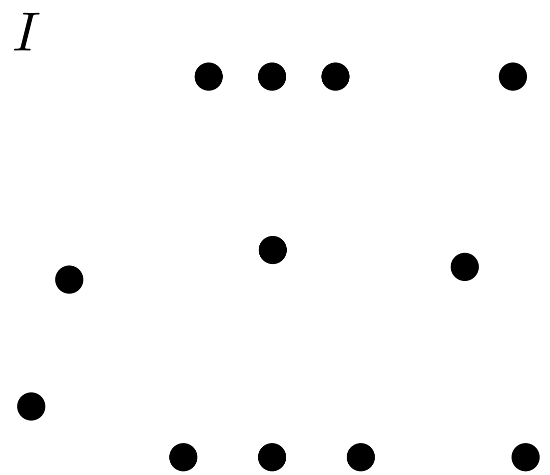

I: a finite groundset

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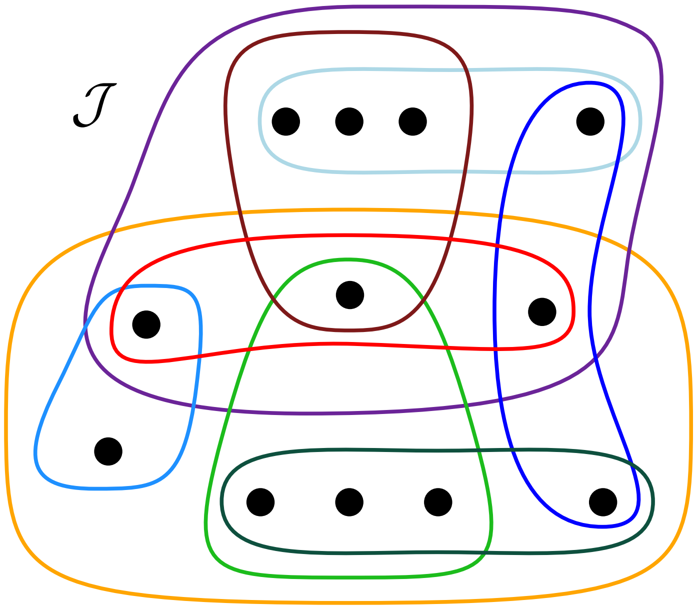

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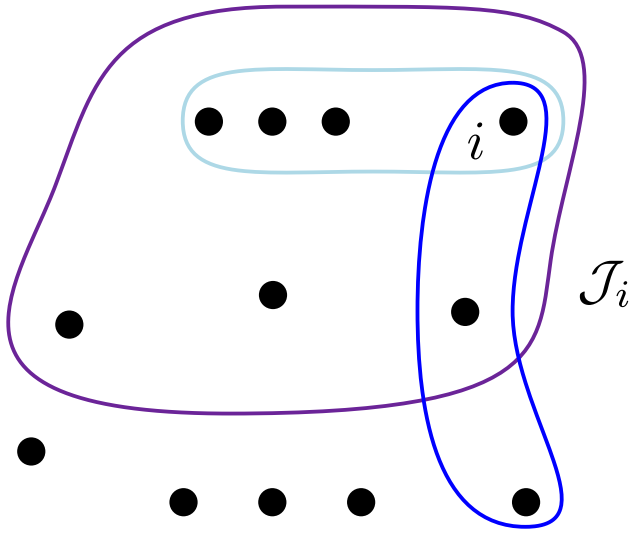

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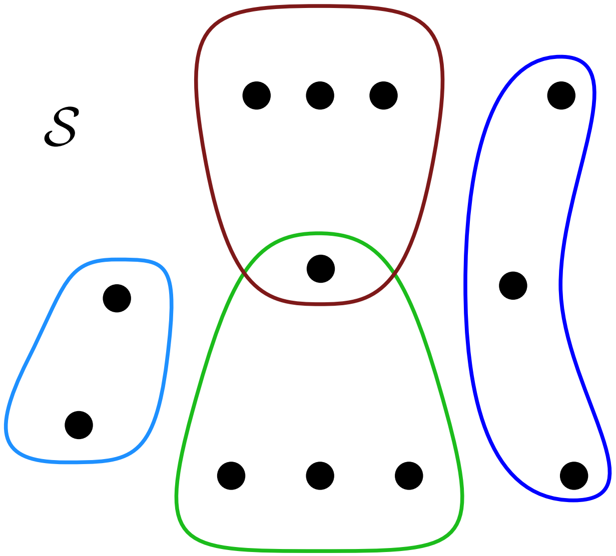

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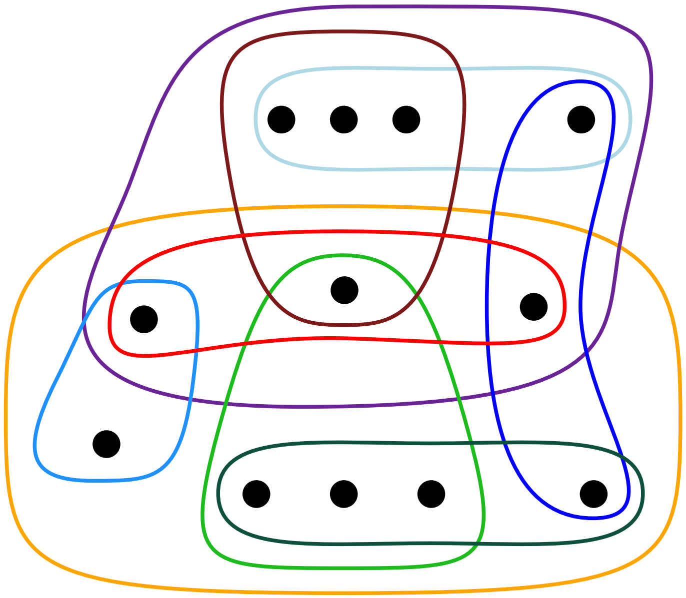
$\mathcal{J}_i \subseteq \mathcal{J}$: subcollection of the subsets covering an element $i \in I$

cover $\mathcal{S}$: a subcollection indexed by $S \subseteq J$ such that $\bigcup_{j \in S} I_j = I$

Set Covering problems

Classical Set Covering Problem (SCP):

Given an instance $(I, \mathcal{J})$ and a cost $c_j \in \mathbb{R}$ for each $j \in J$,



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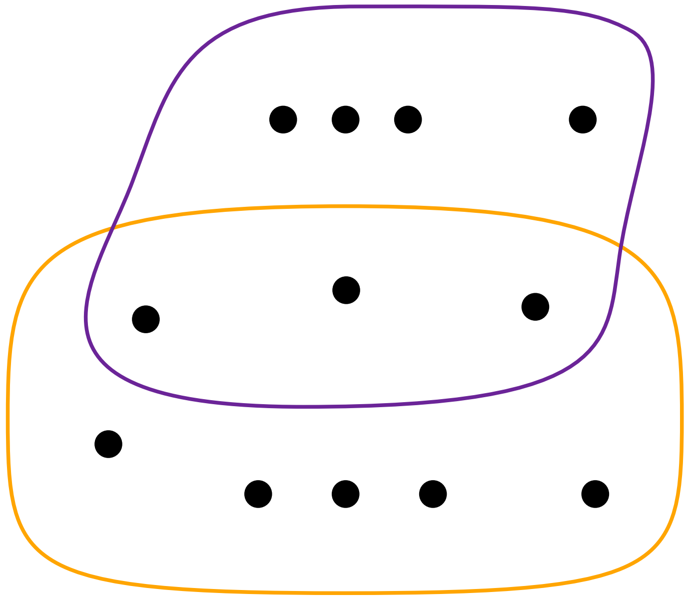

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Variants:	**Set Partitioning**	forbidden overlap
	Set Multicover	required overlap

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Variants:

	Set Partitioning	forbidden overlap
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Also:

Quadratic objective functions

Maximum coverage

...

Coverage share

Given a covering instance $(I, \mathcal{J})$, a cover $\mathcal{S}$ and an element $i \in I$

coverage share: $$r(\mathcal{S}, i) = \frac{1}{1 + |N_i(\mathcal{S})|}$$

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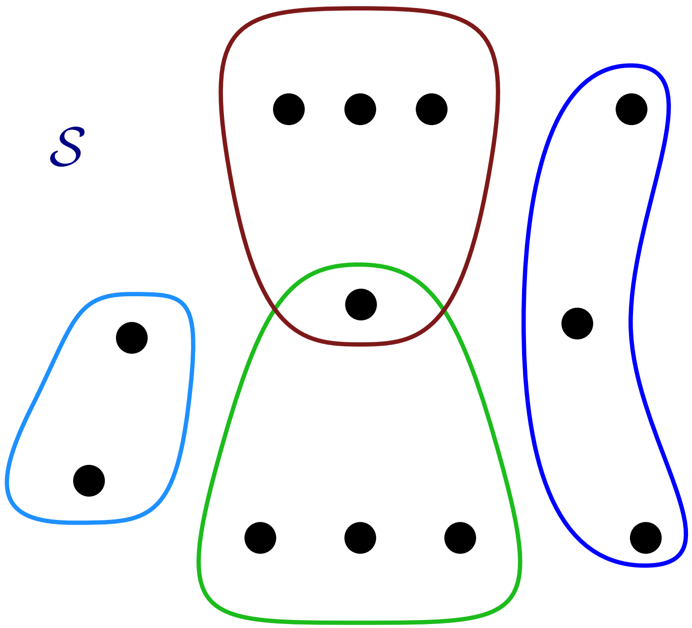

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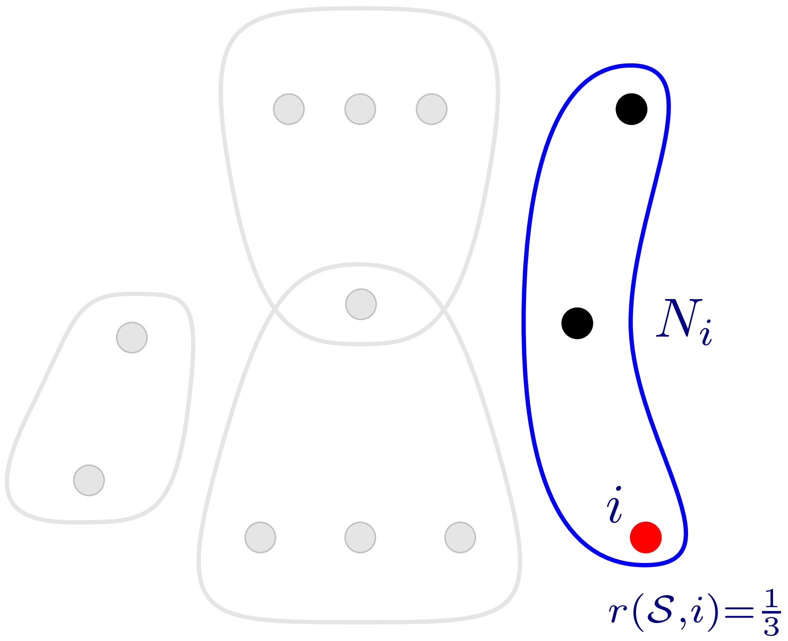

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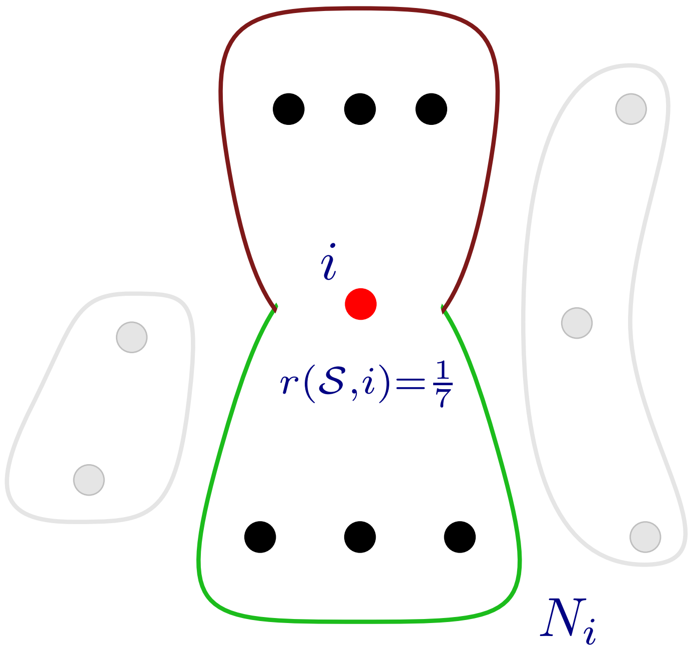

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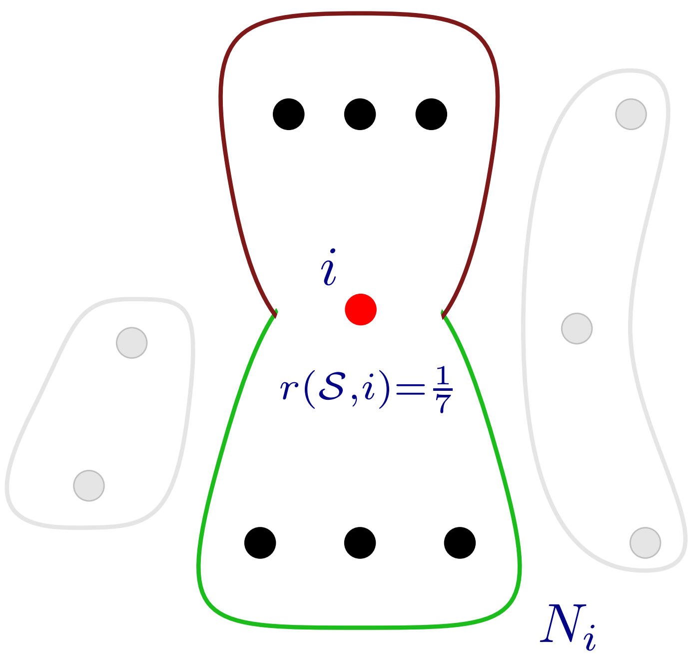

Fraction of resource received by i assuming fair allocation among the competing elements (neighbors of i)

Coverage share problems

Maximum Total Coverage Share Problem (TCSP):

Given an instance $(I, \mathcal{J})$, find a cover $\mathcal{S}$ that maximizes

$$f_t(\mathcal{S}) = \sum_{i \in I} \frac{1}{1 + |N_i(\mathcal{S})|}$$

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Set covering problems with competing ground-set elements

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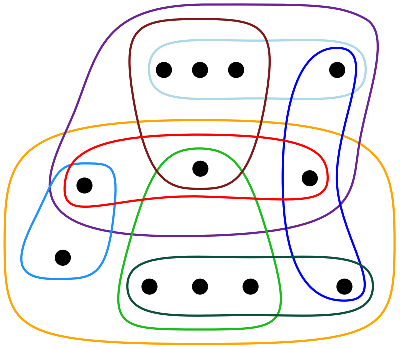
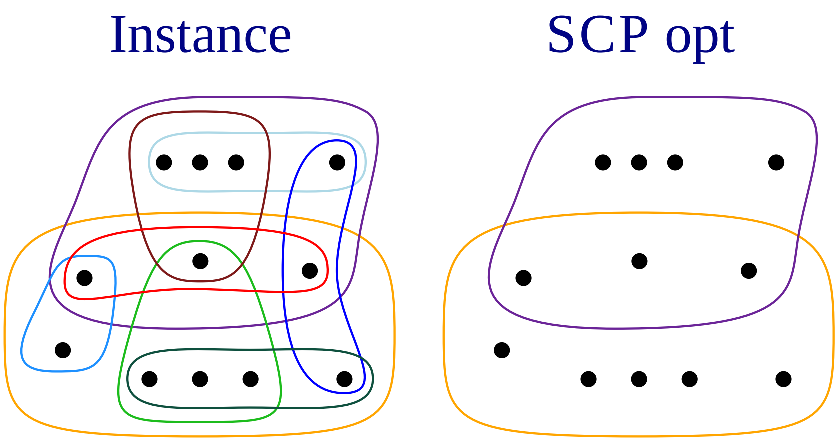
Instance



Coverage share problems

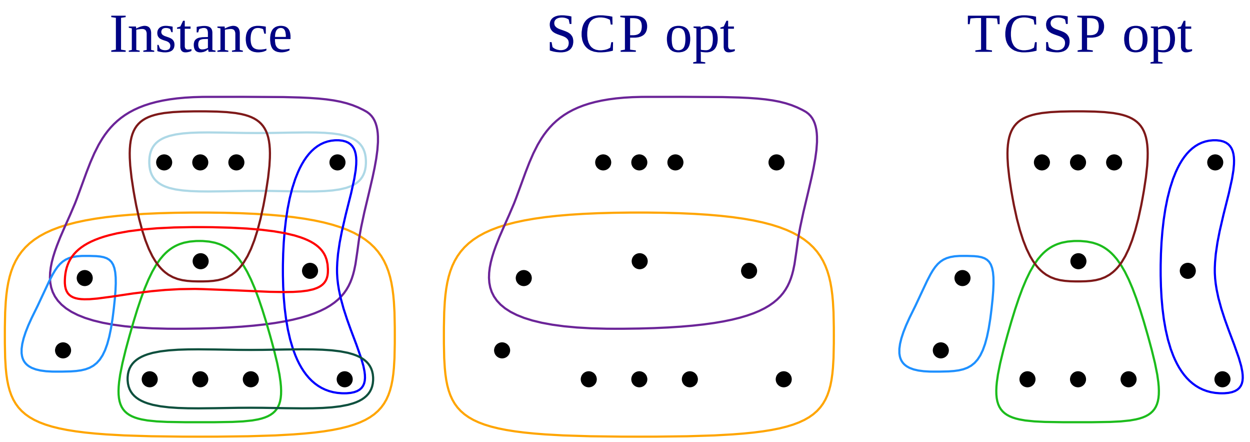


$$|\mathcal{S}| = 2$$

$$f_t(\mathcal{S}) = 1.40$$

$$f_m(\mathcal{S}) = \frac{1}{11}$$

Coverage share problems

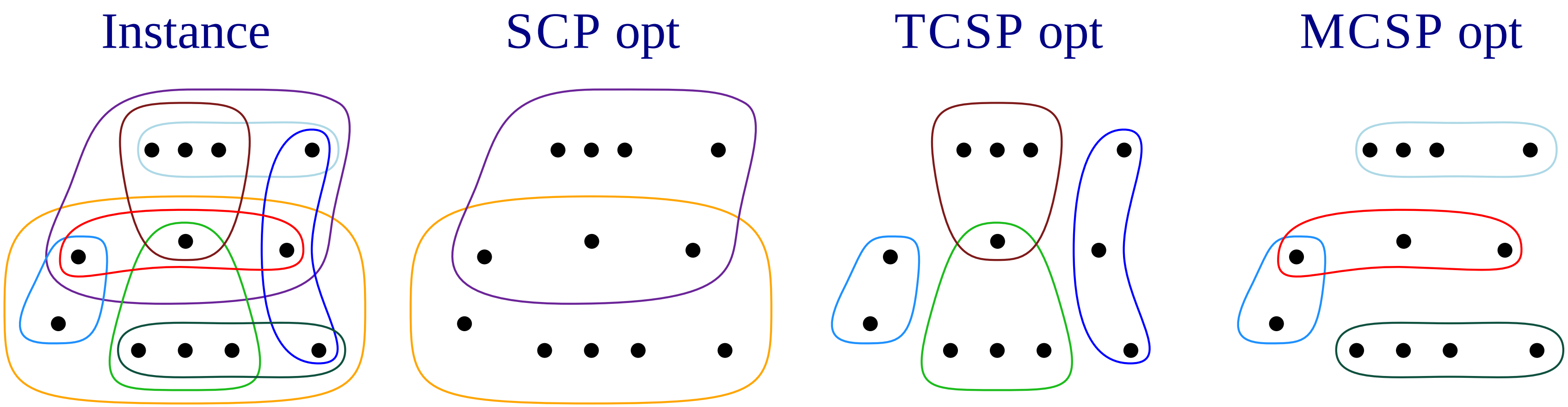


$$|\mathcal{S}| = 2 \qquad |\mathcal{S}| = 4$$

$$f_t(\mathcal{S}) = 1.40 \qquad f_t(\mathcal{S}) = 3.64$$

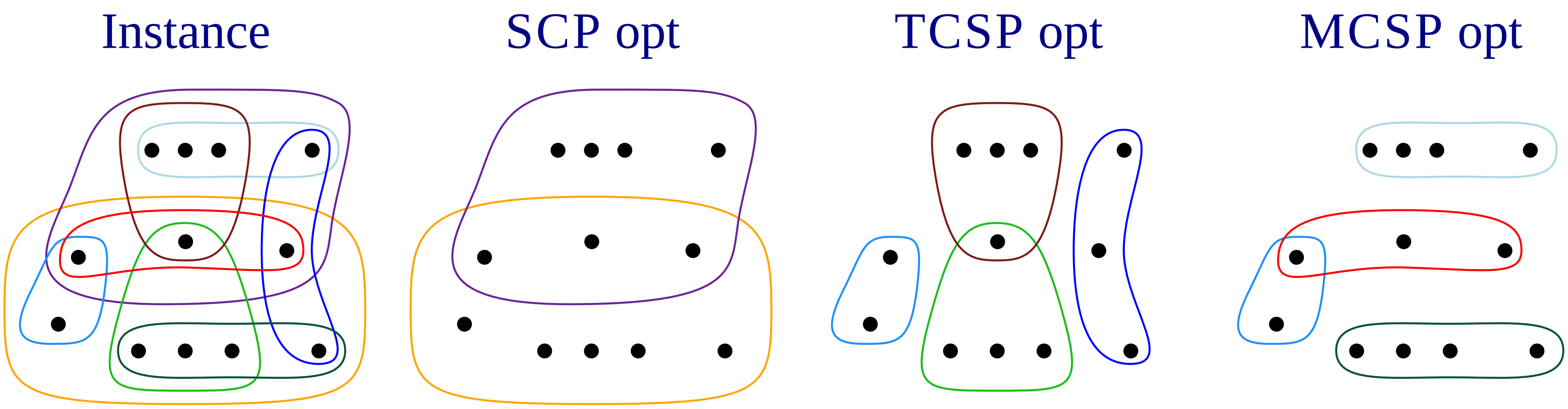
$$f_m(\mathcal{S}) = \frac{1}{11} \qquad f_m(\mathcal{S}) = \frac{1}{7}$$

Coverage share problems



	SCP opt	TCSP opt	MCSP opt
$\lvert\mathcal{S}\rvert$	2	4	4
$f_t(\mathcal{S})$	1.40	3.64	3.42
$f_m(\mathcal{S})$	$\frac{1}{11}$	$\frac{1}{7}$	$\frac{1}{4}$

Coverage share problems



SCP opt:

$$\begin{aligned} |\mathcal{S}| &= 2 \\ f_t(\mathcal{S}) &= 1.40 \\ f_m(\mathcal{S}) &= \frac{1}{11} \end{aligned}$$

TCSP opt:

$$\begin{aligned} |\mathcal{S}| &= 4 \\ f_t(\mathcal{S}) &= 3.64 \\ f_m(\mathcal{S}) &= \frac{1}{7} \end{aligned}$$

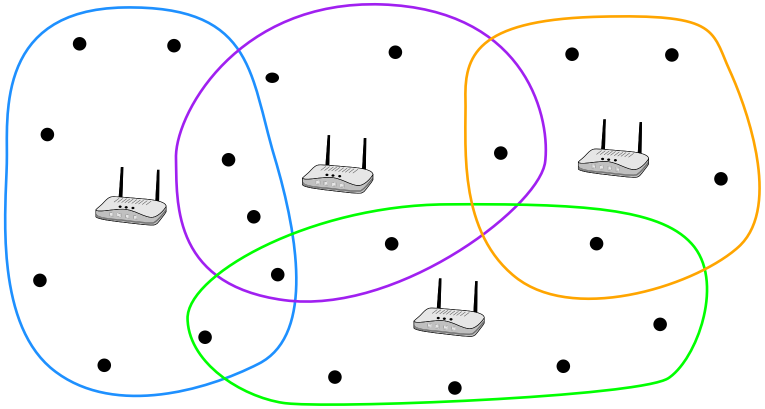
MCSP opt:

$$\begin{aligned} |\mathcal{S}| &= 4 \\ f_t(\mathcal{S}) &= 3.42 \\ f_m(\mathcal{S}) &= \frac{1}{4} \end{aligned}$$

Privilege covers whose subsets have small cardinality and limited overlaps.

Wireless Local Area Network

IEEE 802.11 WLAN: a set of Access Points each able of serving a set of users



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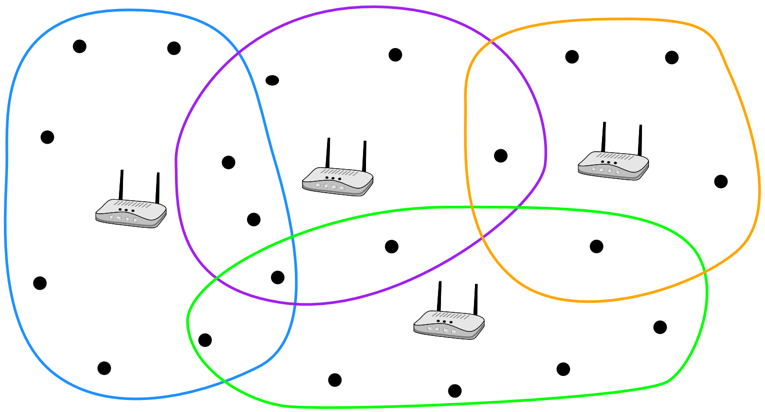

WLANs are becoming pervasive in airports, trains and train stations, private companies, universities, hotels, ...

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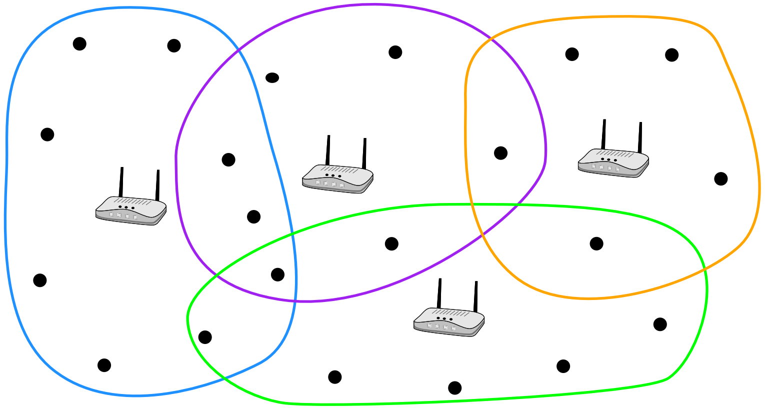

Medium Access Control (MAC) Protocol:

A user can access the network if and only if no other user is interfering directly or indirectly

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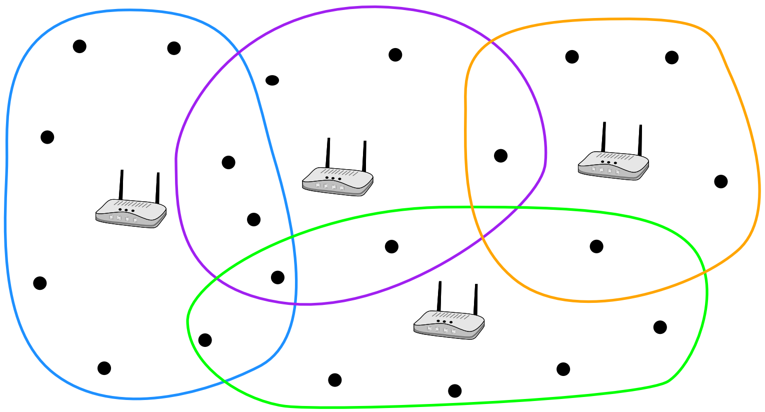

Medium Access Control (MAC) Protocol:

A user can access the network if and only if no other user is interfering directly or indirectly

Assuming uniform peak traffic and fair access after collision,
coverage share of element $i \approx$ fraction of time used by user i

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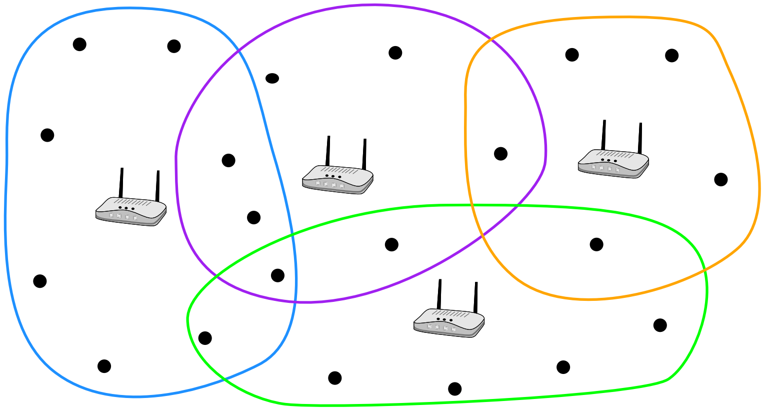

Due to protocol issues, increasing sizes of deployed WLANs and limited resources,

and optimization models and methods can be
very useful to support the planning decisions.

Previous and related work

WLAN design:

- Large-scale WLAN design (Hills 01, ...)
- Max average signal quality in test points (Rodrigues, Mateus and Loureiro 00/01)
- Max coverage level (Kamenetsky and Unbehaun 02)
- Max capacity based on constraint satisfaction (Prommak et al. 02)
- ...
- First hyperbolic model and heuristics (Amaldi, Capone, Cesana and Malucelli 04)

Integer programming formulations

$$\max \quad \sum_{i \in I} \frac{1}{1 + |N_i(\mathcal{S})|}$$

$$(\text{TCSP}) \qquad s.t. \quad \bigcup_{j \in S} I_j = I \qquad \text{complete coverage}$$

$$\mathcal{S} \subseteq \mathcal{J} \qquad \text{select subcollection}$$

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Variables:

$x_j = 1$ if subset I_j is selected (0 otherwise)

$y_{ih} = 1$ if elements i and h are neighbors (0 otherwise)

Integer programming formulations

$$
(\text{TCSP}) \quad
\begin{aligned}
\max \quad & \sum_{i \in I} \frac{1}{1 + \sum_{h \in N_i} y_{ih}} & \\
s.t. \quad & \sum_{j \in J_i} x_j \geq 1 & i \in I \\
& y_{ih} \geq x_j & i \in I,\ h \in N_i,\ j \in J_i \cap J_h \\
& y_{ih} \leq \sum_{j \in J_i \cap J_h} x_j & i \in I,\ h \in N_i \\
& x_j \in \{0, 1\} & j \in J \\
& y_{ih} \in \{0, 1\} & i \in I, h \in N_i
\end{aligned}
$$

Integer programming formulations

$$\max \sum_{i \in I} \frac{1}{1 + \sum_{h \in N_i} y_{ih}} \qquad \rightarrow \quad \text{0-1 hyperbolic sum problem}$$

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Integer programming formulations

$$\max \quad \min_{i \in I} \frac{1}{1 + \sum_{h \in N_i} y_{ih}} \quad = \quad \frac{1}{1 + \min_{i \in I} \max \sum_{h \in N_i} y_{ih}}$$

(MCSP)

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Connection with Quadratic SCP

Quadratic Set Covering Problem (QSCP):

Given $(I, \mathcal{J})$, $Q = \{q_{j\ell} \in \mathbb{R} : j, \ell \in J\}$ (wlog symmetric with zero diagonal) and $\boldsymbol{c} = \{c_j \in \mathbb{R} : j \in J\}$, find a cover $\mathcal{S} \subseteq \mathcal{J}$ that maximizes

$$q(\mathcal{S}) = \frac{1}{2} \sum_{j \in S} \sum_{\ell \in S} q_{j\ell} + \sum_{j \in S} c_j$$

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Choice:

$$c_j = \sum_{i \in I_j} \frac{1}{|I_j|} \qquad q_{j\ell} = \sum_{i \in I_j \cap I_\ell} \left(\frac{1}{|I_j \cup I_\ell|} - \frac{1}{|I_j|} - \frac{1}{|I_\ell|} \right) \quad (\text{for } j \neq \ell)$$

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Then we can verify that:

$f_t(\mathcal{S}) = q(\mathcal{S})$ if at most two subsets overlap

$f_t(\mathcal{S}) \geq q(\mathcal{S})$ otherwise (overhestimated penalty)

Previous and related work

Unconstrained 0-1 Hyperbolic Programming

- Single-ratio: NP-hard, poly with positive denominator (Hammer and Rudeanu 68)
- Multiple-ratio: NP-hard; tackled by SA, Tabu, and decomposing into independent polynomial single-ratio problems (Hansen, Poggi de Aragao and Ribeiro 90/91)

Constrained 0-1 Hyperbolic Programming

- Single-ratio (Stancu-Minasian 97)
- Multiple-ratio: MILP convex reformulations (Tawarmalani, Ahmed and Sahinidis 02)

Quadratic Set Covering Problem

- Various application oriented works (Bazaara 75, Boros, Hammer et al. 00, ...)
- Generic: not $2^{p(|I|)}$-approximable for any polynomial $p()$ Convex: approximable within $O(\ln^2(|I|))$ but not within $\rho \ln^2(|I|)$ (Escoffier and Hammer 05)

Complexity and Approximability (TCSP)

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Generic instances

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Generic instances

- Strongly NP-hard (Amaldi et al. 04)
- Not approximable within $\rho\left(\sqrt{|I|}\right)^{1-\varepsilon}$ or $\rho\left(\sqrt{|\mathcal{J}|}\right)^{1-\varepsilon}$ for a given $\rho > 0$ and any $\varepsilon > 0$, unless $\mathrm{NP} = \mathrm{ZPP}$

  Reduction from Max Independent Set

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  Reduction from Max Independent Set

Euclidean 2D Instances

- Strongly NP-hard (does not admit a FPTAS unless $\mathrm{P} = \mathrm{NP}$)

  Adapting and extending a reduction for Disc-Cover (Fowler et al. 81)
- Under a reasonable restriction, admits a PTAS

  Using the shifting lemma

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Euclidean 1D Instances (or instances with C1C covering matrix)

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  Using the shifting lemma

Euclidean 1D Instances (or instances with C1C covering matrix)

- Polynomial-time solvable

  Longest path on an appropriate directed acyclic digraph

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- Polynomial-time solvable if $|I_j| = 2$
  Reduction to perfect b-matching

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Euclidean 2D Instances

- Strongly NP-hard (does not admit a FPTAS unless P $=$ NP)
  Adapting the reduction for TCSP
- Not approximable within $3/2$ unless P $=$ NP
  Consequence of the above reduction
- Under a reasonable restriction, approximable within a factor 3
  Tiling with hexagons

Linearization

For each ratio $\dfrac{1}{1+\sum_{h\neq i} y_{ih}}$ is introduced a variable $r_i \geq 0$ and the quadratic constraint

$$r_i = \frac{1}{1+\sum_{h\neq i} y_{ih}} \qquad \equiv \qquad r_i + \sum_{h\neq i} r_i y_{ih} = 1$$

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$r_i \cdot y_{ih}$ is standardly linearized with a variable $z_{ih} \geq 0$ and the constraints

$$z_{ih} \leq u_i y_{ih}$$

$$z_{ih} \geq l_i y_{ih}$$

$$z_{ih} \geq r_i + u_i(y_{ih} - 1)$$

$$z_{ih} \leq r_i + l_i(y_{ih} - 1)$$

Linearization

For each ratio $\frac{1}{1+\sum_{h\neq i} y_{ih}}$ is introduced a variable $r_i \geq 0$ and the quadratic constraint

$$r_i = \frac{1}{1+\sum_{h\neq i} y_{ih}} \qquad \equiv \qquad r_i + \sum_{h\neq i} r_i y_{ih} = 1$$

$r_i \cdot y_{ih}$ is standardly linearized with a variable $z_{ih} \geq 0$ and the constraints

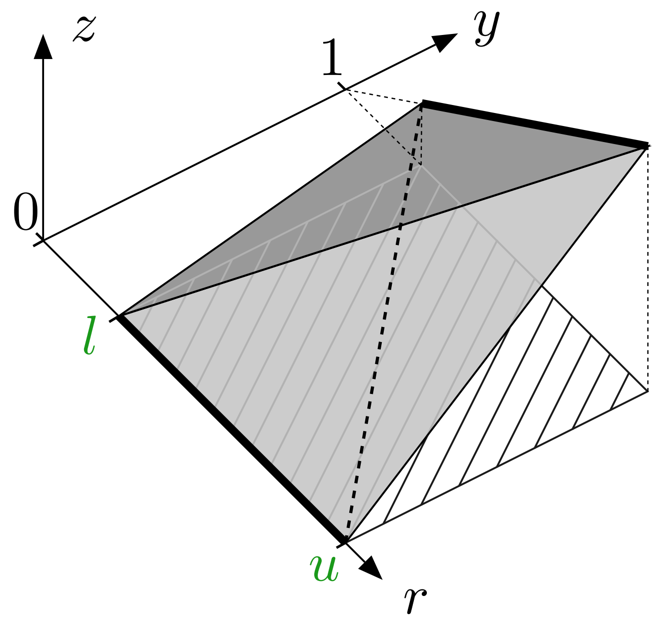
$$z_{ih} \leq u_i y_{ih}$$
$$z_{ih} \geq l_i y_{ih}$$
$$z_{ih} \geq r_i + u_i(y_{ih} - 1)$$
$$z_{ih} \leq r_i + l_i(y_{ih} - 1)$$

NB: r_i is continuous and bounded, and y is binary

Tightening linearization of bilinear terms

$$Z = \{(r, y, z) \;:\; z = r \cdot y, r \in [l, u],\; y \in \{0, 1\}\}$$

$$conv(Z) = \left\{ \begin{array}{ll} z \geq ly, & z \geq r + u(y-1), \\ z \leq uy, & z \leq r + l(y-1) \end{array} \right\}$$



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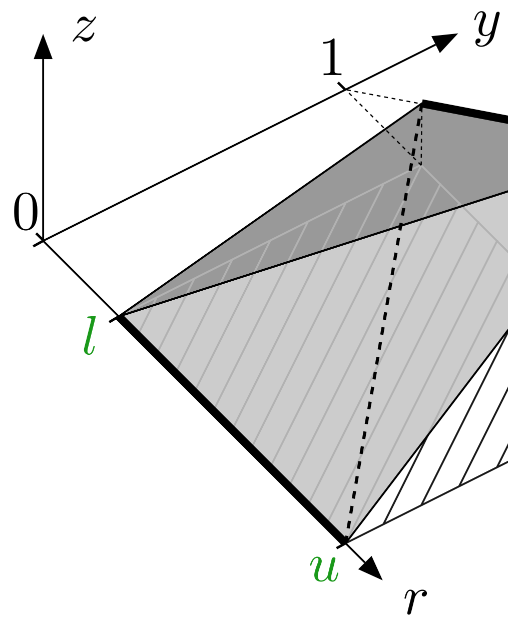

$$\begin{aligned} Z \ = \ & \{(r,y,z) \ : \ z = 0, r \in [l \ \ , u \ \ ], \ y = 0\} \\ \cup \ & \{(r,y,z) \ : \ z = r, r \in [l \ \ , u \ \ ], \ y = 1\} \end{aligned}$$

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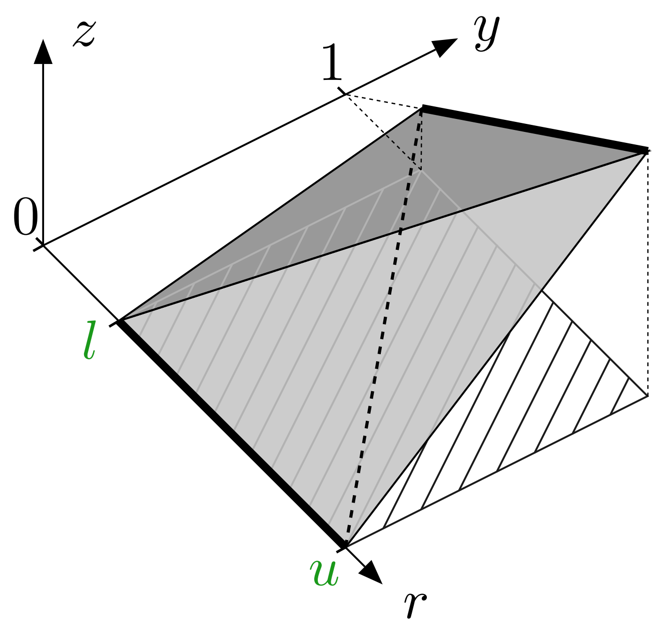

$$Z' = \{(r, y, z) \ : \ z = 0, r \in [l^0, u^0], \ y = 0\}$$
$$\cup \ \{(r, y, z) \ : \ z = r, r \in [l^1, u^1], \ y = 1\}$$

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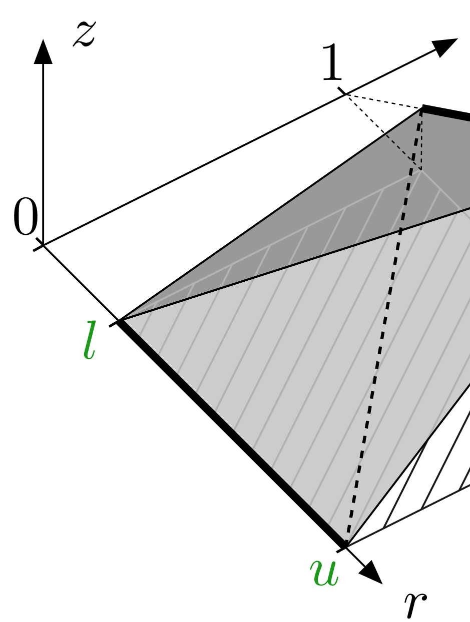

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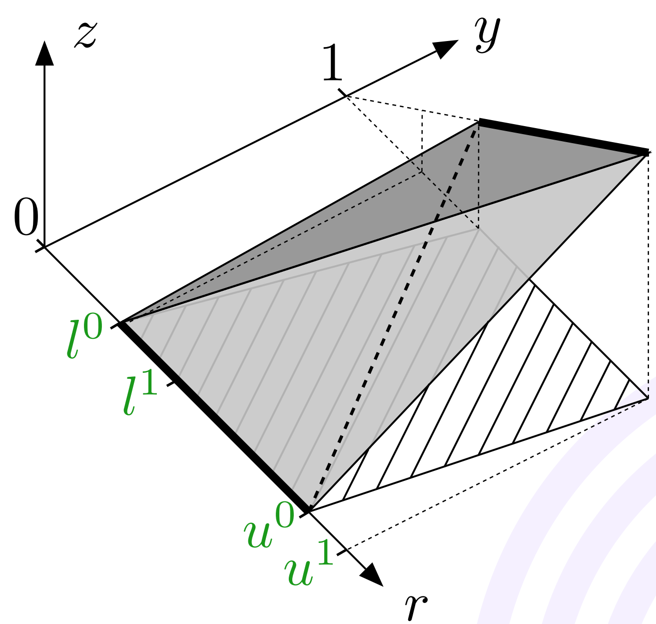

Lagrangean relaxation

By applying Lagragean relaxation to an appropriate reformulation the problem is decomposed into smaller and easier subproblems.

Lagrangean relaxation

By applying Lagragean relaxation to an appropriate reformulation the problem is decomposed into smaller and easier subproblems.

Expanded formulation obtained by adding for each $i \in I$ a vector $\boldsymbol{\chi}_i = \{\chi_{ij} : j \in J_i\}$ of binary variables, one for each covering subset.
Incidence vector of a local covering solution for i.

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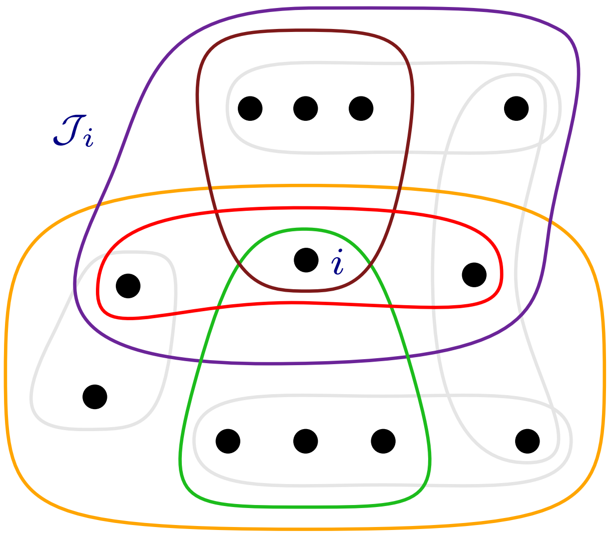

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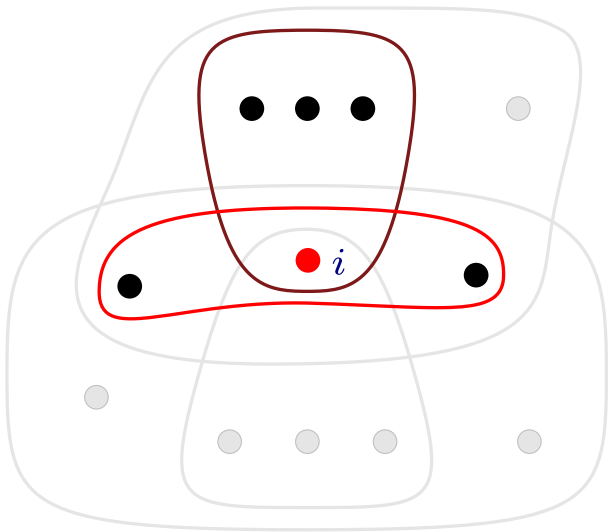

Lagrangean relaxation

Expanded formulation:

$$\max \quad \sum_{i \in I} \frac{1}{1 + \sum_{h \in N_i} y_{ih}}$$

$$\begin{array}{lll} s.t. & \sum_{j \in J_i} x_j \geq 1 & i \in I \\ & y_{ih} \geq x_j & i \in I,\ h \in N_i,\ j \in J_i \cap J_h \\ \hline & \sum_{j \in J_i} \chi_{ij} \geq 1 & i \in I \\ & y_{ih} \geq \chi_{ij} & i \in I,\ h \in N_i,\ j \in J_i \cap J_h \\ & y_{ih} \leq \sum_{j \in J_i \cap J_h} \chi_{ij} & i \in I,\ h \in N_i \\ \hline & x_j = \chi_{ij} & i \in I, j \in J_i \\ & y_{ih} = y_{hi} & i \in I,\ h \in N_i \ :\ h > i \\ \hline & x_j, \chi_{ij}, y_{ih} \in \{0, 1\} & \end{array}$$

Several possibilities, depending on which constraints are deleted/dualized

Lagrangean relaxation

Expanded formulation:

$$
\begin{array}{lll}
\max & \sum_{i \in I} \frac{1}{1 + \sum_{h \in N_i} y_{ih}} & \\
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\end{array}
$$

Without (2), (6) and (7): one SCP and $|I|$ independent hyperbolic subproblems

Lagrangean relaxation

Expanded formulation:

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$$s.t. \quad \sum_{j \in J_i} x_j \geq 1 \qquad i \in I \qquad ($$

$$($$

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LAG$_a$: remove (2) and dualize (6) and (7) $\longrightarrow$ NP-hard hyperbolic subproblems

Lagrangean relaxation

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LAG$_b$: remove (5), (6) and dualize (2), (7) $\quad \longrightarrow \quad$ polynomial hyperbolic subproblems

Lagrangean subproblem for LAG_b

Problem for a given element i:

$$\begin{array}{lll} \max & \dfrac{1}{1+\sum\limits_{h\in N_i} y_h} + \sum\limits_{h\in N_i} c_h y_h & \\ s.t. & \sum\limits_{j\in J_i} \chi_j \geq 1 & \\ & y_h \geq \chi_j & h \in N_i,\ j \in J_i \cap J_h \\ & \chi_j \in \{0,1\} & j \in J_i \\ & y_h \in \{0,1\} & h \in N_i \end{array}$$

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\end{aligned}
$$

Lagrangean subproblem for LAG_b

Fix one variable χ_ℓ to 1 (try all). This covers all $h \in I_\ell$.

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Fix all other χ_j to 0 (no o.f. contribution).

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Lagrangean subproblem for LAG_b

Remains an unconstrained problem with hyperbolic+linear o.f..

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Since hyperbolic depends only on how many and not on which:

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1) sort c_h coefficients in nonincreasing order.

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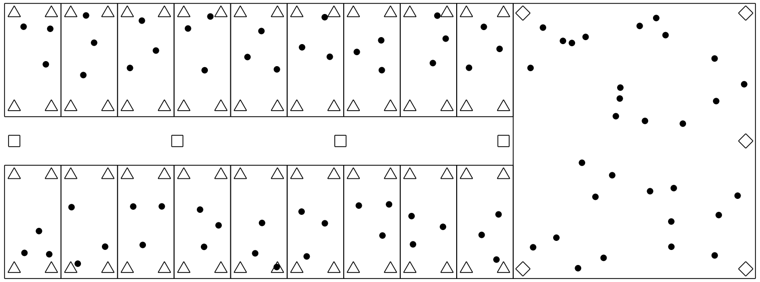
Since hyperbolic depends only on how many and not on which:

1) sort c_h coefficients in nonincreasing order.

2) fix the first k variables to 1, the remaining to 0 (try all k).

Comparison - our department



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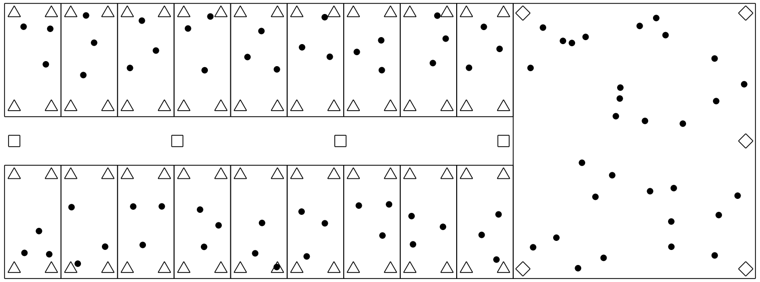

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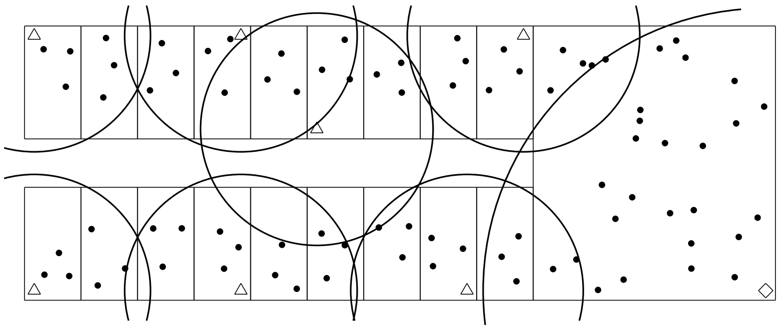
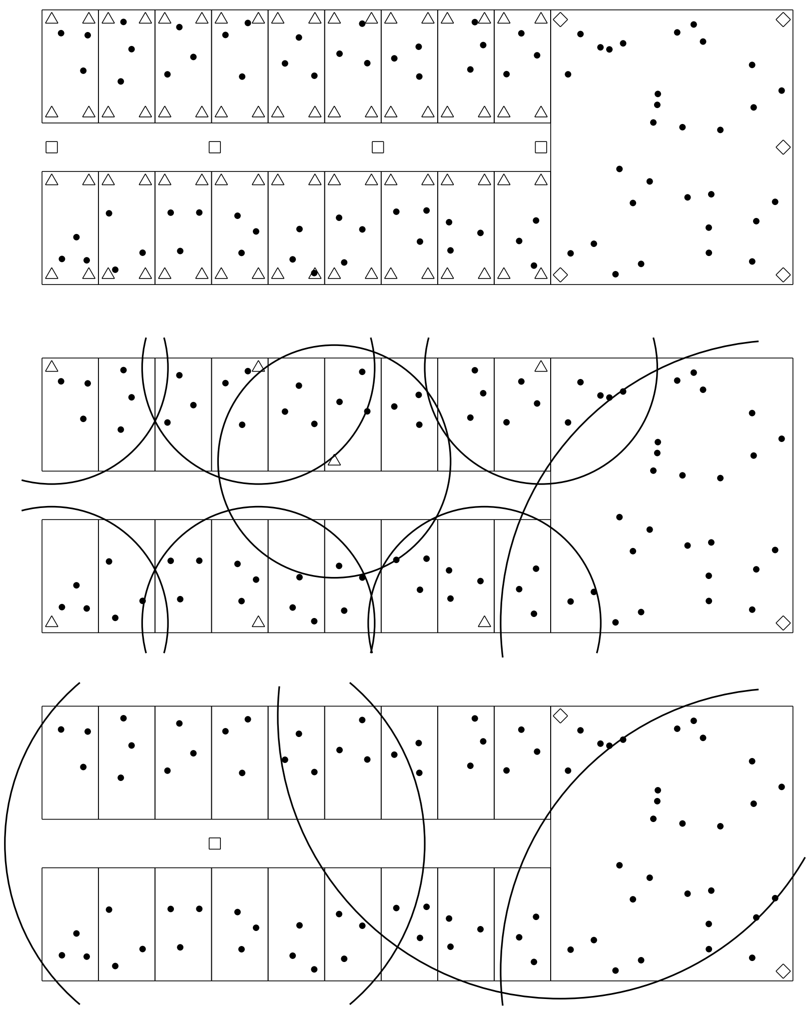


TCSP best solution

			Standard Linearization		Improved Linearization		LAG_b	
$\|J\|$	$\|I\|$	den	gap (%)	time (sec)	gap (%)	time (sec)	gap (%)	time (sec)
81	84	13.7	9.28	—	5.08	—	0.87	237.2

— : time limit exceeded

Comparison - our department

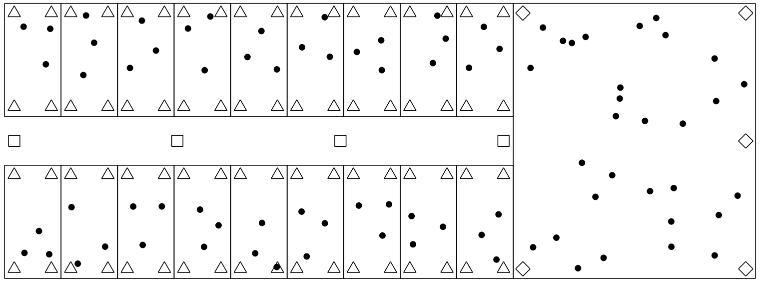


Our Department

TCSP best solution

SCP optimal solution

Comparison - our department



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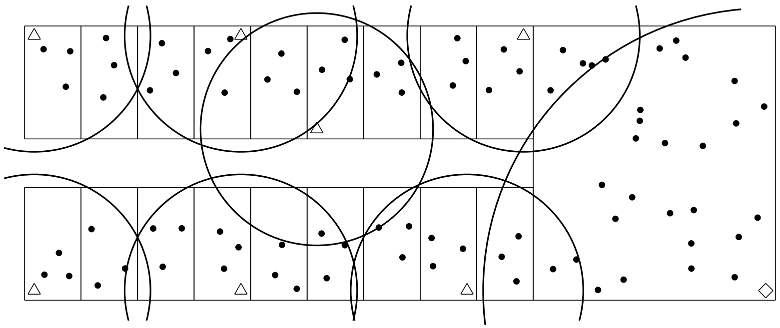

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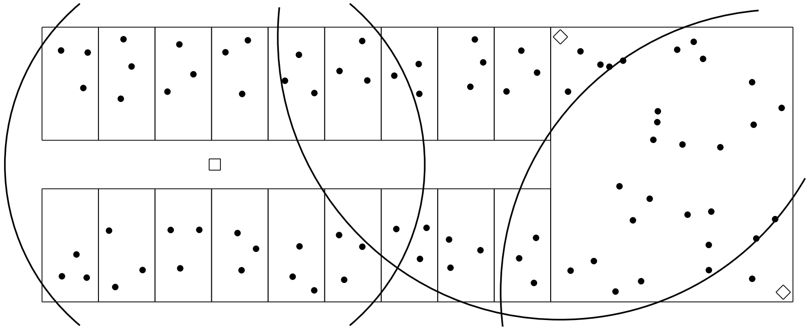

SCP optimal solution

Tests with a WLAN simulator (ns-2):

2.58 Mb/s for SCP solution, 15.8 Mb/s for TCSP solution

Comparison - synthetic instances

$\|J\|$	$\|I\|$	den	Standard Linearization gap	stdev	time	stdev	Improved Linearization gap	stdev	time	stdev	LAG_b gap	stdev	time	stdev
GEOMETRIC INSTANCES (LOW DENSITY)														
50	50	6.6	∗		0.4	0.3	∗		0.1	0.1	∗		0.5	0.3
50	100	6.4	∗		7.2	4.1	∗		4.1	4.1	0.11	0.15	5.3	2.8
100	100	5.3	0.79	1.51	1826.6	1639.3	∗		177.1	203.1	0.28	0.28	24.8	13.6
100	200	5.1	9.42	2.57	–		3.07	1.88	3363.0	530.0	0.37	0.22	409.6	129.5
50	300	6.3	∗		561.6	557.5	∗		395.3	353.9	∗		621.8	137.9
GEOMETRIC INSTANCES (HIGH DENSITY)														
50	50	10.5	∗		15.5	20.3	∗		6.1	5.7	0.17	0.32	2.0	1.7
50	100	10.3	∗		798.6	452.7	∗		313.6	188.9	0.10	0.14	52.5	29.8
100	100	10.8	27.76	3.42	–		12.26	3.43	–		1.89	0.61	215.0	24.8
100	200	10.6	33.17	3.50	–		18.96	1.08	–		2.26	1.14	1111.2	44.5
50	300	11.1	27.12	7.28	–		26.43	6.69	–		0.98	0.69	1494.4	140.7
STANDARD SCP INSTANCES (CLASS SCP4*)														
1000	200	2.0	11.53	1.69	–		6.13	1.65	–		0.12	0.09	3304.6	403.6
STANDARD SCP INSTANCES (CLASS SCPE*)														
500	50	20.0	71.22	4.69	–		35.75	16.62	–		6.14	0.78	1765.9	148.1

∗ : the primal-dual gap is zero (proven optimality)

– : time limit exceeded for all instances of the class

Concluding remarks

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- New interesting class: set covering problems with competing ground-set elements

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Ongoing work:

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